

# The free splitting complex of a free group II: Loxodromic outer automorphisms

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## 1 Introduction

Consider a group  $G$  acting by isometries on a Gromov hyperbolic metric space  $X$ . An element  $g \in G$  is *loxodromic* if for some (any)  $x \in X$  the orbit map  $n \mapsto g^n \cdot x$  is a quasi-isometric embedding  $\mathbb{Z} \mapsto X$ . The terminology comes from the case of hyperbolic 3-space where such an isometry leaves invariant a “loxodromic curve” on the 2-sphere at infinity. The action of a loxodromic  $g$  on the Gromov closure  $\overline{X} = X \cup \partial X$  is a north–south action with attracting–repelling fixed point pair  $\partial_{\pm}g = \lim_{n \rightarrow \pm\infty} g^n \cdot x$ . Two loxodromic elements  $g, g' \in G$  are said to be *coaxial* if the unordered fixed point pairs  $\{\partial_{\pm}g\}$ ,  $\{\partial_{\pm}g'\}$  are equal, and *independent* if those pairs are disjoint. Understanding loxodromic behavior is important, for example, in proving the Tits alternative by “hyperbolic ping-pong” arguments, and for studying second bounded cohomology (see e.g. [BF02]).

Before stating our results, here are some examples. A Gromov hyperbolic group acts on its Cayley graph, each element is either finite order or loxodromic, any two loxodromic elements are either co-axial or independent, and if  $g \in G$  is loxodromic then one has equality of stabilizer subgroups  $\text{Stab}(\partial_-g) = \text{Stab}(\partial_+g) \equiv \text{Stab}(\partial_{\pm}g)$  and this subgroup is virtually cyclic [Gro87].

The mapping class group  $\text{MCG}(S)$  of a finite type surface  $S$  acts on its curve complex  $\mathcal{C}(S)$ , hyperbolicity of which was proved by Masur and Minsky [MM99]. A mapping class  $\phi \in \text{MCG}(S)$  acts loxodromically on  $\mathcal{C}(S)$  if and only if  $\phi$  is pseudo-Anosov, which occurs if and only if  $\phi$  has infinite order and does not preserve any simplex of  $\mathcal{C}(S)$ . Two loxodromics are either co-axial or independent, and for a single loxodromic  $\phi$  the subgroup  $\text{Stab}(\partial_{\pm}\phi)$  is virtually cyclic. These properties are proved first on the level of the stable and unstable lamination pair  $\Lambda_{\phi}^s, \Lambda_{\phi}^u$ , and are then transferred to  $\mathcal{C}(S)$  by showing that for pseudo-Anosov  $\phi, \psi \in \text{MCG}(S)$  one has  $\partial_+\phi = \partial_+\psi$  if and only if  $\Lambda_{\phi}^u = \Lambda_{\psi}^u$ .

$\text{Out}(F_n)$  acts on the free factor complex  $\mathcal{FF}(F_n)$ , hyperbolicity of which was proved by Bestvina and Feighn [BF11], and Theorem 9.3 of that paper proves that  $\phi \in \text{Out}(F_n)$  acts loxodromically on  $\mathcal{FF}(F_n)$  if and only if  $\phi$  is fully irreducible (and see Remark 4.22), which occurs if and only if  $\phi$  has no periodic simplices in  $\mathcal{FF}(F_n)$ . Also, two fully irreducibles are either co-axial or independent, and  $\text{Stab}(\partial_{\pm}\phi)$  is virtually cyclic when  $\phi$  is fully irreducible. Again these properties are related to attracting/repelling lamination pairs: the corresponding properties for laminations were proved in [BFH97]; and from [BF11] it follows that two fully irreducibles are co-axial on  $\mathcal{FF}(F_n)$  if and only if they have the same lamination pair.

**Overview of results.**  $\text{Out}(F_n)$  acts on the free splitting complex  $\mathcal{FS}(F_n)$ , hyperbolicity of which was proved in Part I of this work [HM13e]. Here we study the loxodromic elements for this action, characterizing loxodromic behavior in terms of attracting/repelling laminations and similarly characterizing elements with bounded orbits and with a periodic point (Theorem 1.1). We also prove the same “co-axial versus independent” dichotomy as in all of the above examples (Theorem 1.2). But there are some interesting features of this study which depart from the above examples. One feature (Theorem 1.1) is that there are many more loxodromics acting on  $\mathcal{FS}(F_n)$  than on  $\mathcal{FF}(F_n)$ . In mapping class groups, for  $\phi \in \text{MCG}(S)$  to be pseudo-Anosov there are two equivalent formulations:  $\phi$  has a stable/unstable lamination pair that fills the surface; and  $\phi$  has irreducible powers. This equivalence breaks down in  $\text{Out}(F_n)$ , yielding two different meanings for “loxodromic”:  $\phi$  is loxodromic on  $\mathcal{FS}(F_n)$  if and only if  $\phi$  has a filling lamination pair; whereas  $\phi$  is loxodromic on  $\mathcal{FF}(F_n)$  if and only if it is fully irreducible, a strictly stronger condition. Another feature (Theorem 1.4) is that when  $\phi \in \text{Out}(F_n)$  acts loxodromically on  $\mathcal{FS}(F_n)$ , the subgroup  $\text{Stab}(\partial_{\pm}\phi)$  need not be virtually cyclic: it can be a higher rank abelian group of linearly growing outer automorphisms; or it can contain a surface mapping class group.

**Statements of results.** See Section 2 for a brief review of attracting/repelling lamination pairs, of free factor systems and free factor supports, and of reducible, irreducible, and fully irreducible outer automorphisms. We let  $\mathcal{L}(\phi)$  denote the set of attracting laminations of  $\phi$ . The laminations of  $\mathcal{L}(\phi)$  and  $\mathcal{L}(\phi^{-1})$  come in pairs  $\Lambda_{\phi}^{+}, \Lambda_{\phi}^{-}$  defined by requiring that they have the same free factor support, and if this is not a proper free factor system then we say that the pair  $\Lambda_{\phi}^{\pm}$  fills  $F_n$ . The notation  $\{\Lambda_{\phi}^{\pm}\} = \{\Lambda_{\phi}^{+}, \Lambda_{\phi}^{-}\}$  refers to the corresponding unordered pair. As mentioned above, if  $\phi$  is fully irreducible then it has a filling lamination pair  $\Lambda_{\phi}^{\pm}$  (the unique element of  $\mathcal{L}^{\pm}(\phi)$ ), but the converse is not true in general.

The following “trichotomy theorem” characterizes which elements are loxodromic, which have bounded orbits and which have a periodic point.

**Theorem 1.1.** *The following holds for all  $\phi \in \text{Out}(F_n)$ .*

- (1) *The action of  $\phi$  on  $\mathcal{FS}(F_n)$  is loxodromic if and only if some element of  $\mathcal{L}(\phi)$  fills.*
- (2) *If the action of  $\phi$  on  $\mathcal{FS}(F_n)$  is not loxodromic then the action has bounded orbits.*
- (3) *The action of  $\phi$  on  $\mathcal{FS}(F_n)$  has a periodic point (in fact a periodic vertex) if and only if the full set of attracting laminations  $\mathcal{L}(\phi)$  does not fill.*

See Example 4.1 for a reducible  $\phi$  that acts loxodromically on  $\mathcal{FS}(F_n)$ , and see Example 4.2 for a  $\phi$  with bounded orbits but without periodic points.

**Theorem 1.2.** *Given  $\phi, \psi \in \text{Out}(F_n)$  and filling lamination pairs  $\Lambda_{\phi}^{\pm} \in \mathcal{L}^{\pm}(\phi)$ , one of the following holds:*

- (1)  *$\{\Lambda_{\phi}^{\pm}\} = \{\Lambda_{\psi}^{\pm}\}$  and  $\{\partial_{\pm}\phi\} = \{\partial_{\pm}\psi\}$ , and so  $\phi, \psi$  are co-axial.*
- (2)  *$\{\Lambda_{\phi}^{\pm}\} \cap \{\Lambda_{\psi}^{\pm}\} = \emptyset$  and  $\{\partial_{\pm}\phi\} \cap \{\partial_{\pm}\psi\} = \emptyset$ , and so  $\phi, \psi$  are independent.*

As an application we have the following result solely about attracting laminations, not referring to any complexes on which  $\text{Out}(F_n)$  acts.

**Corollary 1.3.** *For any  $\phi, \psi \in \text{Out}(F_n)$  and any lamination pairs  $\Lambda_\phi^\pm \in \mathcal{L}^\pm(\phi)$  and  $\Lambda_\psi^\pm \in \mathcal{L}^\pm(\psi)$ , if  $\Lambda_\phi^+ = \Lambda_\psi^+$  then  $\Lambda_\phi^- = \Lambda_\psi^-$ .*

*Proof.* Assuming  $\Lambda_\phi^+ = \Lambda_\psi^+$ , the pairs  $\Lambda_\phi^\pm, \Lambda_\psi^\pm$  have the same free factor support  $\mathcal{F}$  which is the conjugacy class  $[F]$  of some nontrivial free factor  $F$ . When  $\mathcal{F}$  is not proper, i.e. when  $F = F_n$  and these pairs fill, the conclusion follows from Theorem 1.2. When  $\mathcal{F}$  is proper then the corollary reduces to the filling case by replacing  $\phi$  with the restricted outer automorphism  $\phi|_F \in \text{Out}(F)$  ([HM13b], Section 1.1.3).  $\square$

When  $\phi \in \text{Out}(F_n)$  acts loxodromically on  $\mathcal{FS}(F_n)$  with filling lamination pair  $\Lambda_\phi^\pm$ , our next theorem provides a detailed description of  $\text{Stab}(\partial_\pm \phi) = \text{Stab}(\Lambda_\phi^\pm)$  and in particular proves that it is finitely generated. Recall the finite index, torsion free, normal subgroup  $\text{IA}_n(\mathbb{Z}/3) < \text{Out}(F_n)$  which is the kernel of the action on  $H_1(F_n; \mathbb{Z}/3) \approx (\mathbb{Z}/3)^n$ . The homomorphism  $\text{PF}_{\Lambda^+}$  in the statement of the theorem is the restriction to  $\text{IA}_n(\mathbb{Z}/3)$  of the expansion factor homomorphism for  $\Lambda_\phi^+$  that is defined in Section 3.3 of [BFH00].

**Theorem 1.4.** *Suppose that  $\phi \in \text{IA}_n(\mathbb{Z}/3)$  is rotationless, that  $\Lambda_\phi^+ \in \mathcal{L}(\phi)$  is filling and that  $K$  is the kernel of  $\text{PF} = \text{PF}_{\Lambda^+} : \text{Stab}(\Lambda_\phi^+) \cap \text{IA}_n(\mathbb{Z}/3) \rightarrow \mathbb{R}$ . Then there exist compact surfaces  $S_1, \dots, S_m$  with nonempty boundary and a homomorphism*

$$\Theta : K \rightarrow \text{MCG}(S_1) \times \dots \times \text{MCG}(S_m)$$

*whose image has finite index and whose kernel is a finitely generated, abelian group of linearly growing outer automorphisms. In particular,  $K$  is finitely generated.*

Compare [BFH97] Section 2 where it is shown that if  $\phi$  is fully irreducible then the kernel of the expansion factor homomorphism is finite, in which case  $K$  is trivial and  $\text{Stab}(\Lambda_\phi^\pm)$  is virtually cyclic. See Example 5.6 in which  $\text{Image}(\Theta)$  is trivial and  $K$  is a rank 2 abelian subgroup of linearly growing outer automorphisms. And see Example 5.16 in which  $\Theta$  maps  $K$  onto a finite index subgroup of a mapping class group.

**Failure of the acylindrical and WPD properties.** As an application of Theorem 1.4 and of Examples 5.6 and 5.16 we note that the action of  $\text{Out}(F_n)$  on  $\mathcal{FS}(F_n)$  is not acylindrical in the sense of Bowditch [Bow08], nor does it satisfy the weaker condition that each  $\phi \in \text{Out}(F_n)$  acting loxodromically on  $\mathcal{FS}(F_n)$  satisfies the WPD property of Bestvina and Fujiwara [BF02], because those conditions imply that  $\text{Stab}(\partial_\pm \phi)$  is virtually cyclic. On the other hand Bestvina and Feighn show in [BF11] that fully irreducible elements acting on  $\mathcal{FF}(F_n)$  do indeed satisfy WPD; it remains unknown whether the action of  $\text{Out}(F_n)$  on  $\mathcal{FF}(F_n)$  is acylindrical.

**The case of rank 2.** The results of this paper (which are all trivial in rank 1) follow in rank 2 from well known results as follows. The abelianization map  $F_2 \mapsto \mathbb{Z}^2$  induces an isomorphism  $\text{Out}(F_2) \approx \text{GL}_2(\mathbb{Z})$  (see [Vog02] for a reference to Nielsen). There is an equivariant containment  $\mathcal{FS}(F_2) \supset \Gamma$ , where  $\Gamma$  is the Farey graph having vertex set  $\mathbb{Q}$  where vertices  $\frac{p}{q}, \frac{r}{s}$  are connected by an edge whenever  $ps - qr = \pm 1$  (see [CV91]). The graph  $\Gamma$  is Gromov hyperbolic [Man05]. One can show that the Gromov boundary of  $\Gamma$

has a  $GL_2(\mathbb{Z})$ -equivariant bijection with the irrational numbers  $\mathbb{R} - \mathbb{Q}$ , that the elements of  $GL_2(\mathbb{Z})$  acting loxodromically on  $\Gamma$  are exactly the matrices having trace of absolute value  $> 2$ , and that these correspond exactly with the exponentially growing elements of  $\text{Out}(F_2)$  each of which is fully irreducible and has a filling lamination.

**Remarks on the proofs.** The table of contents below gives a guide to the proofs of Theorems 1.1 and 1.4. The proof of Theorem 1.2 draws on techniques from the rest of the paper: one part is proved in Section 4.5 using techniques from the proof of Theorem 1.1; and the remainder is proved in Section 6 by applying Theorem 1.4 and its techniques of proof, and by applying the main result from our work [HM13a] on decomposition of subgroups of  $\text{Out}(F_n)$ .

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## 2 Background

In this section we set notation and provide references to [BFH00], [FH11] and [HM13b] for readers that want further details.

**Marked graphs.** We assume that  $F_n$  has been identified with  $\pi_1(R_n, *)$  where  $R_n$  is the graph with one vertex  $*$  and  $n$  edges. A *marked graph*  $G$  is a graph such that each vertex has valence at least two and such that  $G$  is equipped with a homotopy equivalence  $\rho : R_n \rightarrow G$  called the *marking* on  $G$ . The marking provides an identification of  $\pi_1(G)$  with  $F_n$  that is well defined up to composition with an inner automorphism. Thus conjugacy classes of elements and of subgroups of  $\pi_1(G)$  correspond bijectively to conjugacy classes of elements and of subgroups of  $F_n$ . The action of a homotopy equivalence  $f : G \rightarrow G$  on the fundamental group of  $\pi_1(G)$  induces a well defined outer automorphism of  $\pi_1(G)$  and hence a well defined  $\phi \in \text{Out}(F_n)$ ; we say that  $f : G \rightarrow G$  represents  $\phi$  or is a topological representative of  $\phi$ .

**Subgroup systems. Carrying and meet ( $\sqsubset$  and  $\wedge$ ).** The conjugacy class of a finite rank subgroup  $A < F_n$  is denoted  $[A]$ . If  $A_1, \dots, A_k$  are distinct nontrivial, finite rank subgroups then  $\mathcal{A} = \{[A_1], \dots, [A_k]\}$  is called a *subgroup system*. Each  $[A_i]$  is a *component* of  $\mathcal{A}$ . If  $A_1, \dots, A_k$  are non-trivial free factors and if  $F_n = A_1 * \dots * A_k$  or  $F_n = A_1 * \dots * A_k * B$  for some non-trivial free factor  $B$  then  $\mathcal{A}$  is a *free factor system*. More generally if there exists a minimal  $\mathbb{R}$ -tree action  $F_n \curvearrowright T$  with trivial arc stabilizers such that  $\mathcal{A}$  is the set of conjugacy classes of nontrivial point stabilizers then  $\mathcal{A}$  is a *vertex group system*. Given another subgroup system  $\mathcal{A}' = \{[A'_1], \dots, [A'_l]\}$  we use the notation  $\mathcal{A} \sqsubset \mathcal{A}'$  to mean that for each  $i \in \{1, \dots, k\}$  there exists  $j \in \{1, \dots, l\}$  such that  $A_i$  is conjugate to a subgroup of  $A'_j$ ; we refer to this relation by saying that  $\mathcal{A}$  is *carried by* or *contained in*  $\mathcal{A}'$ , also that  $\mathcal{A}'$  is an *extension* of  $\mathcal{A}$ , or simply that  $\mathcal{A} \sqsubset \mathcal{A}'$  is an *extension*. A *filtration by free factor systems* is a sequence of extensions of free factor systems  $\mathcal{F}_0 \sqsubset \dots \sqsubset \mathcal{F}_K$ . The *meet*  $\mathcal{F}_1 \wedge \mathcal{F}_2$  of two free factor systems  $\mathcal{F}_1$  and  $\mathcal{F}_2$  is the unique maximal free factor system that is contained in both  $\mathcal{F}_1$  and  $\mathcal{F}_2$ . By a version of Grushko's theorem,  $\mathcal{F}_1 \wedge \mathcal{F}_2$  is the set of nontrivial conjugacy classes of subgroups of the form  $[A_1 \cap A_2]$  such that  $[A_1] \in \mathcal{F}_1$ ,  $[A_2] \in \mathcal{F}_2$ . The meet operation on pairs extends to a well-defined operation on any set of free factor systems. To each marked graph  $G$  and subgraph  $K$  there corresponds a free factor system denoted  $\mathcal{F}(K)$  or  $[K]$ , by taking the conjugacy classes of the subgroups of  $\pi_1(G) \approx F_n$  corresponding to the noncontractible components of  $K$ . See section 2.6 of [BFH00], and sections 1.1.2 and 3.1 of [HM13b].

**Lines and free factor support.** The *space of lines*  $\mathcal{B} = \mathcal{B}(F_n)$  is the quotient of  $\partial F^n \times \partial F^n - \Delta$  by transposing coordinates and by letting  $F_n$  act. Each line is realized in each marked graph as a bi-infinite path which is unique modulo orientation. A line is birecurrent if for some (any) such realization, each finite subpath is repeated infinitely often in both directions. A line  $\ell$  is *carried by* a free factor system  $\mathcal{F} = \{[A_1], \dots, [A_k]\}$  if for some (any) marked graph  $G$  with subgraph  $H$  corresponding to  $\mathcal{F}$ , the realization of  $\ell$  in  $G$  is contained in  $H$ . A conjugacy class is carried by  $\mathcal{F}$  if it is represented by an element in one of the  $A_i$ 's, equivalently if the periodic line representing that conjugacy class is carried by  $\mathcal{F}$ . The

*free factor support* of a collection of lines or conjugacy classes is the meet of all free factor systems that carry each element of that collection, equivalently the unique minimal free factor system carrying the entire collection. A set of lines *fills*  $F_n$  if its free factor support is  $\{[F_n]\}$ . For each subgraph  $K$  of a marked graph  $G$  the free factor system  $\mathcal{F}(K)$  is the free factor support of the set of conjugacy classes represented by circuits in  $K$ . See section 2.6 of [BFH00] or section 2.5 of [FH11] or sections 1.1.2 and 1.2.2 of [HM13b].

**Filtration, strata, height.** A *filtration* of a marked graph  $G$  is a nested sequence of subgraphs  $\emptyset = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_N = G$ . For any filtration by free factor systems  $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_N = \{[F_n]\}$  there exists a filtered marked graph denoted as above such that  $[G_i] = \mathcal{F}_i$ . Given a filtration of  $G$  as above and a homotopy equivalence  $f: G \rightarrow G$ , if  $f(G_i) \subset G_i$  for all  $i$  then the filtration is *f-invariant* or just *invariant* if  $f$  is clear from the context. The union of edges contained in  $G_i$  but not  $G_{i-1}$  is a subgraph  $H_i$  called the  $i^{\text{th}}$  *stratum*. If  $f(H_i) \subset H_{i-1}$  then  $H_i$  is a *zero stratum*. When working with *f*-invariant filtrations we will always assume that each stratum  $H_i$  is either a zero stratum or an *irreducible stratum*, meaning that for any two edges  $E$  and  $E'$  in  $H_i$  there exists  $k \geq 1$  such that  $f_{\#}^k(E)$  crosses  $E'$ , and that each irreducible stratum is either NEG (nonexponentially growing) or EG (exponentially growing), the latter meaning that for some (every)  $E$  in  $H_i$  the number of  $H_i$  edges crossed by the path  $f_{\#}^k(E)$  grows exponentially in  $k$ . After passing to a higher positive power and subdividing (unnecessary when  $f$  is a CT, for example), any NEG stratum  $H_i$  consists of a single oriented edge  $E$  that satisfies  $f(E) = E \cdot u$  for some (possibly trivial) path in  $G_{i-1}$ . Also, any EG stratum  $H_r$  has the property that for all sufficiently large  $k$  the  $f_{\#}^k$ -image of each edge in  $H_r$  crosses every edge in  $H_r$  and the number of such crossings grows exponentially in  $k$ . Furthermore, any EG stratum has  $\geq 2$  edges. We sometimes say that an edge is EG if it belongs to an EG stratum and similarly for NEG edges. A path or circuit has *height*  $i$  if it is contained in  $G_i$  but not  $G_{i-1}$ . See subsection 1.5.1 of [HM13b] or subsection 2.6 of [FH11].

**Paths,  $f_{\#}$ , Nielsen paths, splittings.** A *path* in  $G$  is an immersion of a (possibly trivial, infinite or bi-infinite) interval and has endpoints, if any, at vertices. We do not distinguish between paths that differ only by an orientation preserving reparameterization of their domains so a path is determined by its associated edge path and we identify a path with its edge path. A path  $\sigma$  is *crossed* by a path  $\tau$  if either  $\sigma$  or  $\bar{\sigma}$  is a subpath of  $\tau$ . A *circuit* in  $G$  is an immersion of a circle.

All homotopy equivalences are assumed to map vertices to vertices and to restrict to an immersion on each edge of  $G$ . If  $\sigma \subset G$  is a finite path and  $f: G \rightarrow G$  is a homotopy equivalence, then  $f(\sigma)$  is homotopic rel endpoints to a unique path that we denote  $f_{\#}(\sigma)$ . Note that  $f_{\#}$  can be iterated and that  $(f_{\#})^k = (f^k)_{\#}$ . If  $f_{\#}^k(\sigma) = \sigma$  for some  $k \geq 1$  then we say that  $\sigma$  is a *periodic Nielsen path*; if  $k = 1$  then  $\sigma$  is a *Nielsen path*. A Nielsen path is *indivisible* if it is not the concatenation of two non-trivial Nielsen subpaths. If  $\tau$  is a circuit, then  $f(\tau)$  is homotopic to a unique circuit  $f_{\#}(\tau)$ .

A decomposition of a path or circuit into subpaths is a *splitting*, written  $\sigma = \sigma_1 \dots \sigma_m$ , if  $f_{\#}^k(\sigma)$  decomposes into subpaths  $f_{\#}^k(\sigma_1) \dots f_{\#}^k(\sigma_m)$  for all  $k \geq 1$ . The “basic splitting property for NEG edges”, often used silently, says that if  $H_i = E$  is an NEG edge then any height  $i$  path splits at the initial endpoint of each occurrence of  $E$  and the terminal



endpoint of each occurrence of  $\bar{E}$ . See section 4 of [BFH00], sections 2.2 and 4.2 of [FH11], Definition 1.27 of [HM13b].

**Rotationless outer automorphisms, CTs, complete splitting.** Every  $\psi \in \text{Out}(F_n)$  has a *rotationless* iterate  $\phi$ , which implies that certain naturally occurring actions of  $\phi$  on finite sets are trivial. As an example, every  $\phi$ -periodic free factor system is fixed by  $\phi$ . It follows that if  $\mathcal{F} \subset \mathcal{F}'$  is an extension of  $\phi$ -invariant free factor systems, and if there is no  $\phi$ -invariant free factor system strictly between  $\mathcal{F}$  and  $\mathcal{F}'$ , then there is no  $\phi$ -periodic free factor system strictly between them; in this case we say that  $\phi$  is *fully irreducible* relative to the extension  $\mathcal{F} \subset \mathcal{F}'$ . Every rotationless iterate is represented by a particularly nice kind of filtered homotopy equivalence  $f : G \rightarrow G$  called a CT which stands for “completely split relative train track representative”; in fact if  $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_K$  is any  $\phi$ -invariant filtration by free factor systems then there is a CT having filtration elements representing each  $\mathcal{F}_i$ . In a CT, a *complete splitting* of a path or circuit is a splitting into terms that are either single edges in irreducible strata, indivisible Nielsen paths, exceptional paths (see below) or certain paths in zero strata. We shall often refer to the defining properties of CTs by their **(Parenthesized Titles)** as found in the citations below. For example, the **(Completely split)** property says that  $f(E)$  is completely split for each edge  $E$  of an irreducible stratum, and similarly for certain paths in zero strata. Also, the property **(Filtration)** says that for each filtration element  $G_i$  of the given filtration  $\emptyset = G_0 \subset G_1 \subset G_2 \subset \dots \subset G_N = G$ , the core subgraph of  $G_i$  is also a filtration element, and if  $[G_{i-1}] \neq [G_i]$  then  $\phi$  is fully irreducible relative to  $[G_{i-1}] \subset [G_i]$ . See sections 3.3 and 4.1 of [FH11] or Definitions 1.27, 1.28 and 1.29 of [HM13b].

**Twist paths, NEG Nielsen paths, exceptional paths, linear families.** Suppose that  $f : G \rightarrow G$  is a CT. An NEG edge  $E$  is *linear* if  $f(E) = E \cdot u$  for some Nielsen path  $u$ . In this case, there is a closed root-free Nielsen path  $w$  such that  $u = w^d$  for some  $d \neq 0$ . The path  $w$  is called the *twist path associated to  $E$*  or sometimes just a *twist path*. Paths of the form  $Ew^p\bar{E}$  are indivisible Nielsen paths and every indivisible Nielsen path with NEG height is of this form. The unoriented conjugacy class determined by  $w$  is called the *axis* or *twistor* associated to  $E$ . If  $E_i$  and  $E_j$  are distinct linear edges with the same axes then  $w_i = w_j$  and  $d_i \neq d_j$ . In this case we say that  $E_i$  and  $E_j$  belong to the same *linear family*. If  $d_i$  and  $d_j$  have the same sign then a path of the form  $E_i w^s \bar{E}_j$  is called an *exceptional path in the linear family associated to  $w$*  or sometimes just an *exceptional path*. See the beginning of section 4.1 of [FH11] or Definition 1.27 or [HM13b].

**Attracting laminations,  $\mathcal{L}(\phi)$ , expansion factor homomorphism PF.**

Given  $\phi \in \text{Out}(F_n)$  its set of *attracting laminations*  $\mathcal{L}(\phi)$  is the finite collection of all sets of lines  $\Lambda \subset \mathcal{B}$  such that  $\Lambda$  is the closure of a single bi-recurrent, nonperiodic line  $\ell \in \Lambda$ , and there is an open set  $U \subset \mathcal{B}$  and  $k \geq 1$  such that  $\{\phi^{ik}(U) \mid i \geq 0\}$  is a neighborhood basis of  $\ell$  (equivalently of  $\Lambda$ ). We call  $U$  an *attracting neighborhood* of  $\ell$ , and we call  $\ell$  a *generic line* of  $\Lambda$ . When  $\phi$  is rotationless we may take  $k = 1$ , in which case there is a bijection between the set of EG strata of any representative CT  $f : G \rightarrow G$  and the set  $\mathcal{L}(\phi)$ . We need two characterizations of this bijection:  $H_i \leftrightarrow \Lambda$  if and only if a generic leaf of  $\Lambda$  is contained in  $G_i$  but not in  $G_{i+1}$ ; and  $H_i \leftrightarrow \Lambda$  if and only if the free factor system

$[G_i]$  properly contains the free factor system  $[G_{i-1}]$  and  $[G_i]$  is the support of the union of the lines in the lamination  $\Lambda$  and the lines carried by  $[G_{i-1}]$ ; the equivalence of these is a direct consequence of the CT defining property (**Filtration**). Also, there is a pairing between elements of  $\mathcal{L}(\phi)$  and the elements of  $\mathcal{L}(\phi^{-1})$  characterized by the property that paired laminations have the same free factor support. See sections 3.1 and 3.2 of [BFH00].

Suppose that  $\Lambda^+ \in \mathcal{L}(\phi)$  corresponds to  $H_r$ . A line  $\ell \in \Lambda^+$  is generic if and only if both of its ends are dense in  $\Lambda^+$  and  $\ell$  is *semi-generic* if exactly one of its ends is dense in  $\Lambda^+$ . Lines that are neither semi-generic nor generic have height  $< r$ . If  $H_r$  is an EG stratum then there is at most one (up to a change of orientation) indivisible Nielsen path of height  $r$ . Let  $u < r$  be the maximal index for which  $H_u$  is irreducible. If there is an indivisible Nielsen path of height  $r$  then  $u = r - 1$ . Otherwise it may be that  $u < r - 1$ , in which case we say that the zero strata  $H_{u+1}, \dots, H_{r-1}$  are *enveloped* by  $H_r$  and write  $H_r^z = \cup_{i=u+1}^r H_i$ . See Definition 2.18 of [FH11] or Definition 1.28 of [HM13b].

Given  $\Lambda \in \mathcal{L}(\phi)$  for some  $\phi \in \text{Out}(F_n)$ , the stabilizer of  $\Lambda$  is denoted  $\text{Stab}(\Lambda) < \text{Out}(F_n)$ . The *asymptotic expansion factor homomorphism* denoted  $\text{PF} = \text{PF}_\Lambda : \text{Stab}(\Lambda) \rightarrow (0, +\infty)$  is well-defined as follows. Choose  $f : G \rightarrow G$  to be any CT representing a rotationless power of  $\phi$ , with EG stratum  $H_r$  corresponding to  $\Lambda$ , and given  $\psi \in \text{Stab}(\Lambda)$  choose  $g : G \rightarrow G$  to be any topological representative of  $\psi$  defined on  $G$ . For each finite path  $\sigma$  in  $G$  let  $EL_r(\sigma)$  denote the number of times  $\sigma$  crosses edges of  $H_r$ . Then  $\text{PF}_\Lambda(\psi)$  is the unique number with the property that for any  $\epsilon > 0$  there exists  $N > 0$  such that for each finite subpath  $\sigma$  of a generic leaf of  $\Lambda$  in  $G$ , if  $EL_r(\sigma) \geq N$  then

$$\left| \text{PF}_\Lambda(\psi) - \log \left( \frac{EL_r(g_\#(\sigma))}{EL_r(\sigma)} \right) \right| \leq \epsilon$$

When  $\Lambda$  is clear from context we write simply  $\text{PF}(\psi)$ . See Section 3.3 of [BFH00] for details.

**PG and UPG.** An outer automorphism  $\phi \in \text{Out}(F_n)$  is polynomially growing or PG if and only if  $\mathcal{L}(\phi) = \emptyset$ . Assuming that  $\phi$  is PG, we say that  $\phi$  is UPG if the action of  $\phi$  on  $H_1(F_n) \approx \mathbb{Z}^n$  is unipotent. Every PG element of  $\text{IA}_n(\mathbb{Z}/3)$  is UPG. See Section 5.7 of [BFH00].

**Multi-edge extensions, and EG strata.** We say that an extension of free factor systems  $\mathcal{F} \sqsubset \mathcal{F}'$  is a *one edge extension* if it is realized in some marked graph  $G$  by a pair of subgraphs  $H \subset H'$  such that  $H' \setminus H$  consists of one edge of  $G$ . Otherwise,  $\mathcal{F} \sqsubset \mathcal{F}'$  is a *multi-edge extension*. Let  $\phi \in \text{Out}(F_n)$  be rotationless. If  $\phi$  is fully irreducible relative to a properly nested extension  $\mathcal{F} \sqsubset \mathcal{F}'$  of  $\phi$ -invariant free factor systems then the following hold: if  $\mathcal{F} \sqsubset \mathcal{F}'$  is a one-edge extension then every attracting lamination carried by  $\mathcal{F}'$  is carried by  $\mathcal{F}$ ; whereas if  $\mathcal{F} \sqsubset \mathcal{F}'$  is a multi-edge extension then there exists a unique attracting lamination  $\Lambda \in \mathcal{L}(\phi)$  that is carried by  $\mathcal{F}'$  but not by  $\mathcal{F}$ . See [BFH00] Section 3.1 and Corollary 3.2.2.

**Weak attraction and the nonattracting subgroup system  $\mathcal{A}_{\text{na}}(\Lambda^\pm)$ .** Consider a rotationless  $\phi \in \text{Out}(F_n)$  and a lamination pair  $\Lambda^\pm$  for  $\phi$ . Choose a CT  $f : G \rightarrow G$  representing  $\phi$ . If the circuit  $\sigma \subset G$  represents the conjugacy class  $[a]$  then  $[a]$  is *weakly attracted*



to  $\Lambda^+$  if for each finite subpath  $\gamma$  of some (every) generic leaf of  $\Lambda^+$  (realized in  $G$ ) there exists  $K = K(\gamma)$  such that  $\gamma$  is a subpath of  $f_{\#}^k(\sigma)$  for all  $k \geq K$ . Weak attraction of lines is defined similarly. Weak attraction is well-defined independent of the choice of CT. The *nonattracting subgroup system* for  $\Lambda^+$ , denoted  $\mathcal{A}_{\text{na}}(\Lambda^+)$ , is the unique vertex group system with the property that a conjugacy class  $[c]$  of  $F_n$  is not weakly attracted to  $\Lambda^+$  if and only if there exists  $[A] \in \mathcal{A}_{\text{na}}(\Lambda^+)$  such that  $c$  is conjugate to an element of  $A$ . If  $\Lambda^- \in \mathcal{L}(\phi^{-1})$  is paired with  $\Lambda^+$  then  $\mathcal{A}_{\text{na}}(\Lambda^+) = \mathcal{A}_{\text{na}}(\Lambda^-)$  so we usually write  $\mathcal{A}_{\text{na}}(\Lambda^{\pm})$ . Although not indicated in the notation, the definition of  $\mathcal{A}_{\text{na}}(\Lambda^{\pm})$  depends on  $\phi$  as well as  $\Lambda^{\pm}$ . See section 6 of [BFH00] and Section 1 of [HM13c].

**Geometricity of EG strata and attracting laminations.** We review here certain basic properties of geometricity; see Section 5 for a more in depth review. Given a CT  $f: G \rightarrow G$ , its EG strata are classified as being either *geometric* or *non-geometric*. Given  $\phi \in \text{Out}(F_n)$ , its attracting laminations  $\Lambda \in \mathcal{L}(\phi)$  are also classified as geometric or non-geometric, in that exactly one of the following holds: *either* for every CT  $f: G \rightarrow G$  representing a rotationless power of  $\phi$ , the EG stratum corresponding to  $\Lambda$  is geometric; *or* for every such CT the EG stratum corresponding to  $\Lambda$  is nongeometric. An EG stratum  $H_r$  is geometric if and only if there exists a closed, indivisible Nielsen path  $\rho$  of height  $r$ , if and only if the nonattracting subgroup system  $\mathcal{A}_{\text{na}}(\Lambda_r)$  is *not* a free factor system; also,  $\rho$  is the unique indivisible height  $r$  Nielsen path up to reversal. Furthermore, if  $H_r$  and  $\Lambda_r$  are geometric, and if  $H_r$  is the top stratum, then  $\mathcal{A}_{\text{na}}(\Lambda_r)$  consists of the free factor system  $[G_{r-1}]$  and one additional rank 1 component  $[\langle \rho \rangle]$ . See Section 5.3 of [BFH00] and Section 2 of [HM13b].

### 3 Free splittings and marked graph pairs

A *free splitting* of  $F_n$  is a minimal simplicial action of  $F_n$  on a simplicial tree  $T$  with trivial edge stabilizers; we follow the convention of suppressing the action and letting  $T$  stand for the free splitting. Two free splittings are equivalent if there is an equivariant homeomorphism between their trees. The homeomorphism is not assumed to be simplicial so the equivalence class of a free splitting is completely determined by its natural simplicial structure; i.e. the one in which all vertices have valence at least three. Note that if two free splittings have the same underlying trees and if their  $F_n$  actions differ only by conjugation by an element of  $F_n$  then the free splittings are equivalent. A *k-edge splitting* is one with  $k$  orbits of natural edges.

Given a marked graph  $G$  with universal cover  $\tilde{G}$ , the marking on  $G$  provides an identification of the group of covering translation of  $\tilde{G}$  with  $F_n$  that is well defined up to composition with an inner automorphism and so determines a well defined equivalence class of free splittings. Every free splitting for which the action is proper occurs in this manner, and equivalence classes of proper free splittings correspond precisely to equivalence classes of their quotient marked graphs.

**Remark 3.1.** One can understand free splittings up to equivalence by studying an appropriate quotient object. For example, associated to a free splitting  $T$  is its quotient graph of groups  $T/F_n$  [SW79]. The fundamental group  $\pi_1(T/F_n)$  is defined in the category of

graphs of groups [Ser80], and  $\pi_1(T/F_n)$  is identified with  $F_n$  up to conjugacy, which provides a “marking” for  $T/F_n$ . We could therefore understand free splittings as marked graphs of groups up to appropriate equivalence. Rather than pursue this line of thought, we avoid the concept of  $\pi_1(T/F_n)$  altogether and instead pursue the more topological approach of a “marked graph pair”, which is very close to a “graph of spaces” as used in [SW79].

A homotopy equivalence  $h : (G, \rho) \rightarrow (G', \rho')$  between marked graphs *preserves markings* if  $h\rho$  is homotopic to  $\rho'$ . Equivalently, there is a lift  $\tilde{h} : \tilde{G} \rightarrow \tilde{G}'$  that is equivariant with respect to the  $F_n$ -actions on  $\tilde{G}$  and  $\tilde{G}'$ . Two marked graphs  $(G, \rho)$  and  $(G', \rho')$  are *equivalent* if there is a homeomorphism  $h : G \rightarrow G'$  that preserves markings. Equivalently,  $\tilde{G}$  and  $\tilde{G}'$  determine equivalent free splittings.

If  $H$  is a subgraph of  $G$  then we write  $G - H$  for the complement of  $H$  in  $G$  and  $G \setminus H$  for the closure of  $G - H$ . Thus  $G \setminus H$  is the subgraph that is the union of all edges not contained in  $H$ .

A *marked graph pair* is a pair  $(G, H)$  where  $G$  is a marked graph and  $H$  is a natural subgraph of  $G$ , all of whose components are non-contractible. The number of natural edges in  $G \setminus H$  is the *co-edge number* of  $(G, H)$ .

**Definition 3.2.** Define a relation on marked graph pairs by  $(G, H) \sim (G', H')$  if there is a homotopy equivalence  $h : (G, H) \rightarrow (G', H')$  such that

- (1)  $h : G \rightarrow G'$  preserves markings.
- (2)  $h$  induces a bijection between the natural vertices of  $G - H$  and the natural vertices of  $G' - H'$ .
- (3) there is a bijection  $E \longleftrightarrow E'$  between the edges of  $G \setminus H$  and the edges of  $G' \setminus H'$  so that for each  $E$ ,  $h(E) = \mu' E' \nu'$  where  $\mu'$  and  $\nu'$  are paths that are either trivial or contained in  $H'$ .

**Lemma 3.3.** *The relation of Definition 3.2 is an equivalence relation on marked graph pairs.*

*Proof.* The reflexive and transitive properties are clear so it suffices to assume that  $h : (G, H) \rightarrow (G', H')$  satisfies (1)–(3) and produce  $g : (G', H') \rightarrow (G, H)$  that satisfies (1)–(3).

Since  $h$  restricts to a homotopy equivalence from  $H$  to  $H'$ , it induces a bijection  $H_i \longleftrightarrow H'_i = h(H_i)$  between the components of  $H$  and the components of  $H'$ . Lift  $h$  to an equivariant map of universal covers  $\tilde{h} : \tilde{G} \rightarrow \tilde{G}'$  and let  $\tilde{H} \subset \tilde{G}$  and  $\tilde{H}' \subset \tilde{G}'$  be the full pre-images of  $H$  and  $H'$  respectively. Each component of  $\tilde{H}$  is a copy  $\tilde{C}_i$  of the universal cover of an  $H_i$ , and similarly for components of  $\tilde{H}'$ . The map  $\tilde{h}$  induces a bijection between the components of  $\tilde{H}$  and of  $\tilde{H}'$ , so that  $\tilde{C}_i \leftrightarrow \tilde{C}'_i$  if and only if  $\tilde{C}_i, \tilde{C}'_i$  have the same stabilizer subgroup in  $F_n$ .

Construct an equivariant map  $\tilde{g} : (\tilde{G}', \tilde{H}') \rightarrow (\tilde{G}, \tilde{H})$  as follows. For each natural vertex  $v' \in G'$  choose a lift  $\tilde{v}' \in \tilde{G}'$ . If  $v' \in G' - H'$  then there is a unique natural vertex  $v \in G - H$  such that  $h(v) = v'$  and hence a unique natural vertex  $\tilde{v} \in \tilde{G}$  such that  $\tilde{h}(\tilde{v}) = \tilde{v}'$ . Define  $\tilde{g}(\tilde{v}') = \tilde{v}$ . Otherwise,  $\tilde{v}'$  is contained in a component  $\tilde{C}'_i$  of  $\tilde{H}'$  and we define  $\tilde{g}(\tilde{v}') = \tilde{v}$  where  $\tilde{v}$  is an arbitrary natural vertex in the corresponding  $\tilde{C}_i$ . Having defined  $\tilde{g}$  on one vertex in each  $F_n$ -orbit of natural vertices, now extend it equivariantly over all natural vertices and

then extend it equivariantly over all natural edges so that it is injective on each edge. Note that  $\tilde{g}$  and  $\tilde{h}$  induce the same bijection between the components of  $\tilde{H}$  and the components of  $\tilde{H}'$ .

The induced homotopy equivalence  $g : (G', H') \rightarrow (G, H)$  preserves markings and induces the same bijection between the natural vertices of  $G' - H'$  and the natural vertices of  $G - H$  as  $h$ . Thus (1) and (2) are satisfied. For each natural edge  $\tilde{E}$  of  $\tilde{G} \setminus \tilde{H}$  there are paths  $\tilde{\mu}', \tilde{\nu}'$  that are either trivial or contained in  $\tilde{H}'$  and a natural edge  $\tilde{E}'$  of  $\tilde{G}' \setminus \tilde{H}'$  such that  $\tilde{h}(\tilde{E}) = \tilde{\mu}' \tilde{E}' \tilde{\nu}'$ ; this defines a bijection  $\tilde{E} \longleftrightarrow \tilde{E}'$  that projects to the bijection  $E \longleftrightarrow E'$  associated to  $h$ . Let  $\tilde{v}_1$  and  $\tilde{v}_2$  be the initial and terminal endpoints of  $\tilde{E}$  respectively and let  $\tilde{v}'_1$  and  $\tilde{v}'_2$  be the initial and terminal endpoints of  $\tilde{E}'$  respectively. If  $\tilde{v}'_1 \in \tilde{G}' - \tilde{H}'$  then  $\tilde{g}(\tilde{v}'_1) = \tilde{v}_1$  and we take  $\tilde{\mu}$  to be the trivial path based at  $\tilde{v}_1$ . Otherwise  $\tilde{h}(\tilde{v}_1)$  and  $\tilde{v}'_1$  are the endpoints of  $\tilde{\mu}'$  and so belong to the same component of  $\tilde{H}'$ . It follows that  $\tilde{g}(\tilde{v}'_1)$  and  $\tilde{v}_1$  belong to the same component of  $\tilde{H}$  and so bound a path  $\tilde{\mu} \subset \tilde{H}$ . Similarly, either  $\tilde{g}(\tilde{v}'_2) = \tilde{v}_2$  and  $\tilde{\nu}$  is trivial or  $\tilde{g}(\tilde{v}'_2)$  and  $\tilde{v}_2$  bound a path  $\tilde{\nu} \subset \tilde{H}$ . By construction,  $\tilde{g}(\tilde{E}') = \tilde{\mu} \tilde{E} \tilde{\nu}$ . This completes the proof that  $g$  satisfies (3) and so completes the proof of the lemma.  $\square$

An equivalence class of marked graphs determines an equivalence class of free splittings. The same is true for an equivalence class of marked graph pairs. The following lemma shows that every free splitting arises from this construction.

**Lemma 3.4.** *For each equivalence class  $[(G, H)]$  of marked graph pairs, the equivalence class  $\langle G, H \rangle$  of free splittings obtained from  $\tilde{G}$  by collapsing each component of  $\tilde{H}$  to a point is well defined. Moreover,  $[(G, H)] \longleftrightarrow \langle G, H \rangle$  defines a bijection between the set of equivalence classes of co-edge  $k$  marked graph pairs and the set of equivalence classes of  $k$ -edge free splittings.*

*Proof.* Consider equivalent marked graph pairs  $(G, H)$  and  $(G', H')$  and choose  $h : (G, H) \rightarrow (G', H')$  satisfying (1)–(3) of Definition 3.2. Equip  $\tilde{G}$  and  $\tilde{G}'$  with actions on  $F_n$  that are compatible with the markings and let  $\tilde{h} : \tilde{G} \rightarrow \tilde{G}'$  be an equivariant lift. Then  $\tilde{h}$  induces a bijection between the natural vertices of  $\tilde{G} - \tilde{H}$  and the natural vertices of  $\tilde{G}' - \tilde{H}'$  and a bijection  $\tilde{E} \longleftrightarrow \tilde{E}'$  between the natural edges of  $\tilde{G} \setminus \tilde{H}$  and the natural edges of  $\tilde{G}' \setminus \tilde{H}'$  so that for each  $\tilde{E}$ ,  $\tilde{h}(\tilde{E}) = \tilde{\mu}' \tilde{E}' \tilde{\nu}'$  where  $\tilde{\mu}'$  and  $\tilde{\nu}'$  are paths that are either trivial or contained in  $\tilde{H}'$ . After collapsing each component of  $\tilde{H}$  to a point and each component of  $\tilde{H}'$  to a point,  $\tilde{h}$  induces an equivariant map that is homotopic to an equivariant homeomorphism  $\hat{h} : \tilde{G}/\tilde{H} \rightarrow \tilde{G}'/\tilde{H}'$ . (The induced map itself may fail to be injective on edges because it collapses the subintervals that map to  $\tilde{\mu}'$  or  $\tilde{\nu}'$  to points.) This proves that there is a well defined map  $[(G, H)] \mapsto \langle G, H \rangle$ .

Surjectivity is well known, in that any free splitting  $T$  can be obtained from some properly discontinuous free splitting  $T'$  by collapsing to a point each component of some invariant subforest of  $T'$ .

It remains to show that the map is injective. Suppose that there is an equivariant homeomorphism  $\tau$  between  $T = \tilde{G}/\tilde{H}$  and  $T' = \tilde{G}'/\tilde{H}'$ . We must show that  $(G, H)$  and  $(G', H')$  are equivalent marked graph pairs. For each natural vertex  $v \in G$ , choose a lift  $\tilde{v} \in \tilde{G}$  and let  $\tilde{v}_T$  be its projected image in  $T$ . If  $v \in G - H$  then  $\tilde{v}_T$  has trivial stabilizer so  $\tau(\tilde{v}_T) \in T'$  has trivial stabilizer and lifts to a unique natural vertex  $\tilde{v}' \in \tilde{G}' - \tilde{H}'$  that we define to be  $\tilde{h}(\tilde{v})$ . Otherwise,  $\tilde{v}_T$  is the projected image of some component  $\tilde{H}_i$  of  $\tilde{H}$

and  $\tau(\tilde{v}_T)$  lifts to a component  $\tilde{H}'_i$  of  $\tilde{H}'$ ; in this case define  $\tilde{h}(\tilde{v})$  to be an arbitrary natural vertex in  $\tilde{H}'_i$ . Now extend  $\tilde{h}$  equivariantly over all natural vertices and then equivariantly over all natural edges so that it is injective on each edge.

The induced homotopy equivalence  $h : (G, H) \rightarrow (G', H')$  preserves markings and induces a bijection between the natural vertices of  $G - H$  and the natural vertices of  $G' - H'$ . Thus (1) and (2) of Definition 3.2 are satisfied. There is a bijection  $\tilde{E} \longleftrightarrow \tilde{E}'$  between the set of natural edges  $\tilde{E} \subset \tilde{G} \setminus \tilde{H}$  and the set of natural edges  $\tilde{E}' \subset \tilde{G}' \setminus \tilde{H}'$  with the property that  $\tilde{E}'$  projects to the edge in  $T'$  that is the  $\tau$  image of the projection into  $T$  of  $\tilde{E}$ . Let  $\tilde{v}_1$  and  $\tilde{v}_2$  be the initial and terminal endpoints of  $\tilde{E}$  respectively and let  $\tilde{v}'_1$  and  $\tilde{v}'_2$  be the initial and terminal endpoints of  $\tilde{E}'$  respectively. If  $\tilde{v}_1 \in \tilde{G} - \tilde{H}$  then  $\tilde{h}(\tilde{v}_1) = \tilde{v}'_1$  and we let  $\tilde{\mu}'$  be the trivial path based at  $\tilde{v}'_1$ . Otherwise  $\tilde{h}(\tilde{v}_1)$  and  $\tilde{v}'_1$  belong to the same component of  $\tilde{H}'$  and so bound a path  $\tilde{\mu}' \subset \tilde{H}'$ . Similarly, either  $\tilde{h}(\tilde{v}_2) = \tilde{v}'_2$  and  $\tilde{\nu}'$  is trivial or  $\tilde{h}(\tilde{v}_2)$  and  $\tilde{v}'_2$  bound a path  $\tilde{\nu}' \subset \tilde{H}'$ . By construction,  $\tilde{h}(\tilde{E}) = \tilde{\mu}' \tilde{E}' \tilde{\nu}'$  so  $h$  satisfies item (3) of Definition 3.2 and we are done.  $\square$

An equivariant simplicial map  $f : S \rightarrow T$  between free splittings is a *collapse map* if  $f$  is injective over the interior of each edge of  $T$ ; an edge of  $S$  belongs to the *collapsed subgraph*  $\sigma$  if its  $f$ -image is a vertex. Thus  $\sigma$  is an  $F_n$ -invariant forest and  $T$  is obtained from  $S$  by collapsing each component of  $\sigma$  to a point.

The free splitting complex  $\mathcal{FS}(F_n)$  is a simplicial complex with one  $k$  simplex for each equivalence class of  $(k+1)$ -edge splitting and with the simplex corresponding to  $T$  being a face in the simplex corresponding to  $S$  if there is a non-trivial collapse map  $S \mapsto T$  between the natural structures of  $S$  and  $T$ . Denote the first barycentric subdivision of  $\mathcal{FS}(F_n)$  by  $\mathcal{FS}'(F_n)$ .

In the language of marked graph pairs we have

**Lemma 3.5.** *For any marked graph pair  $(G, H)$  and any proper natural subgraph  $H'$  that properly contains  $H$ , the simplex in  $\mathcal{FS}(F_n)$  determined by  $\langle G, H' \rangle$  is a face of the simplex in  $\mathcal{FS}(F_n)$  determined by  $\langle G, H \rangle$ . Moreover, all the faces of the simplex determined by  $\langle G, H \rangle$  have this form.*

*Proof.* If  $H'$  is a proper natural subgraph of  $G$  that properly contains  $H$  then  $\tilde{H}'$  is a proper natural forest in  $\tilde{G}$  that properly contains  $\tilde{H}$ . The image of  $\tilde{H}'$  in the tree  $\tilde{G}/\tilde{H}$  obtained from  $\tilde{G}$  by collapsing each component of  $\tilde{H}$  to a point is a proper natural forest of  $\tilde{G}/\tilde{H}$  that contains at least one orbit of edges. Collapsing that forest defines a collapse map  $\tilde{G}/\tilde{H} \rightarrow \tilde{G}/\tilde{H}'$ . Since  $\tilde{G}/\tilde{H}$  represents  $\langle G, H \rangle$  and  $\tilde{G}/\tilde{H}'$  represents  $\langle G, H' \rangle$  this proves that the simplex determined by  $\langle G, H' \rangle$  is a face of the simplex determined by  $\langle G, H \rangle$ . The converse follows from the fact that any natural  $F_n$ -invariant forest of  $\tilde{G}/\tilde{H}$ , and in particular any collapsed subgraph of  $\tilde{G}/\tilde{H}$ , projects to a subgraph of  $G \setminus H$ .  $\square$

Suppose that  $T$  is a free splitting and that  $a \cdot x$  denotes the image of  $x \in T$  under the action of  $a \in F_n$ . For any  $\Phi \in \text{Aut}(F_n)$ , define a new free splitting  $T^\Phi$  with the same underlying tree  $T$  by having the image of  $x$  under the new action of  $a$  be  $\Phi(a) \cdot x$ . If  $\Phi_1$  and  $\Phi_2$  determine the same element  $\phi \in \text{Out}(F_n)$  then they differ by an inner automorphism and  $T^{\Phi_1}$  and  $T^{\Phi_2}$  are equivalent free splittings. We therefore have a well defined right action of  $\text{Out}(F_n)$  on  $\mathcal{FS}(F_n)$ . Note that if  $a \in F_n$  acts elliptically on  $T$  then  $\Phi^{-1}(a)$  acts elliptically on  $T^\Phi$ .

We can express this action in the language of marked graph pairs as follows. If  $G$  is a marked graph with marking  $\rho$  and if  $f : G \rightarrow G$  is a homotopy equivalence representing the outer automorphism  $\phi$ , let  $G^f = G^\phi$  be the marked graph with underlying graph  $G$  and marking  $f\rho$ . For any marked graph pair  $(G, H)$ , we denote  $(G^f, H) = (G^\phi, H)$  by  $(G, H)^f = (G, H)^\phi$  and the free splitting that this pair determines by  $\langle G, H \rangle^f = \langle G, H \rangle^\phi$ . Note that for any marked graph pair  $(G, H)$ , and for any homotopy equivalence  $f : G \rightarrow G$ , however  $f$  may change marking on  $G$ , it preserves marking in the context of  $f : G \rightarrow G^f$  and of  $f : (G, H) \rightarrow (G, H)^f$ . This fact is important later in several applications of Definition 3.2.

## 4 Proof of Theorem 1.1: Classification of actions

In this section we prove the first of our three main results.

**Theorem 1.1.** *The following hold for all  $\phi \in \text{Out}(F_n)$ .*

- (1) *The action of  $\phi$  on  $\mathcal{FS}(F_n)$  is loxodromic if and only if some element of  $\mathcal{L}(\phi)$  fills.*
- (2) *If action of  $\phi$  on  $\mathcal{FS}(F_n)$  is not loxodromic then the action has bounded orbits.*
- (3) *The action of some iterate of  $\phi$  on  $\mathcal{FS}(F_n)$  fixes a point (in fact a vertex) if and only if  $\mathcal{L}(\phi)$  does not fill.*

The proofs of the items of this theorem are spread across various subsections to follow.

**Example 4.1.** If  $\phi$  is fully irreducible then the unique element of  $\mathcal{L}(\phi)$  fills so we are in case (1). It is also easy to construct reducible examples in which an element of  $\mathcal{L}(\phi)$  fills. Let  $G$  be a marked graph with an invariant subgraph  $G_1 \subset G$  and unique vertex  $v$  where  $G_1$  is a rose of rank  $m \geq 2$  and  $H_2 = G \setminus G_1$  is a rank two rose with edges  $A$  and  $B$ . Assume that the marking identifies the fundamental groups of  $G_1 < G$  with  $F_m < F_{m+2}$ . Choose a closed path  $\sigma \subset G_1$  based at  $v$  such that the conjugacy class determined by  $\sigma$  fills  $F_m$ . Define  $f : G \rightarrow G$  to be the identity on  $G_1$  and by

$$A \mapsto A\sigma\bar{B}\sigma B \quad B \mapsto B\sigma A\sigma\bar{B}\sigma B$$

and let  $\phi$  be the outer automorphism determined by  $f$ . Then  $f : G \rightarrow G$  is a relative train track map, and  $H_2$  is an EG stratum with an associated lamination  $\Lambda$ . For each  $k \geq 0$ ,  $f_\#^k(B)$  is an initial subpath of  $f_\#^{k+1}(B)$ . The singular ray  $R_B$  determined by  $B$  is the union of the increasing sequence

$$B \subset f_\#(B) \subset f_\#^2(B) \subset \dots$$

The line  $L = R_B^{-1}\sigma R_B$  is a weak limit of the subpaths  $f_\#^k(\bar{B}\sigma B)$  of  $f_\#^{k+1}(B)$  and so is a leaf of  $\Lambda$ . Any free factor  $\mathcal{F}$  that carries  $L$  also carries the line  $L \diamond L = R_B^{-1}\sigma^2 R_B$  obtained by concatenating  $L$  with itself and then tightening. Similarly,  $\mathcal{F}$  carries each  $R_B^{-1}\sigma^m R_B$ . Since  $\sigma$  is a weak limit of these lines,  $\sigma$  is carried by  $\mathcal{F}$ . Thus the smallest free factor  $\mathcal{F}$  that carries  $\Lambda$  properly contains  $[F_m]$ . Corollary 3.32 of [BFH00] implies that  $\phi$  does not preserve any co-rank one free factor that contains  $[F_m]$  so  $\mathcal{F} = [F_{m+2}]$  and  $\Lambda$  fills.

Note that in Example 4.1, if  $\theta \in \text{Out}(F_{m+2})$  is represented by an automorphism  $\Theta$  that fixes each element of the subgroup  $\langle A, B, \sigma \rangle$  then  $\theta$  is represented by a homotopy equivalence

of  $G$  that commutes with  $f$  up to homotopy rel  $v$ . In this case,  $\theta$  commutes with  $\phi$  and so is contained in the stabilizer of  $\Lambda$ ; see Examples 5.6 and 5.16.

**Example 4.2.** We can modify the above example to create an element satisfying the conclusion of (2) but not satisfying (3), that is, a  $\phi \in \text{Out}(F_n)$  acting with bounded orbits on  $\mathcal{FS}(F_n)$  but with no periodic points. Define  $G'$  from  $G$  by adding a third stratum  $H_3 = G' \setminus G$  consisting of two loops  $A'$  and  $B'$  attached to the unique vertex of  $G$ . The marking identifies  $G \subset G'$  with  $F_{m+2} < F_{m+4}$ . Extend  $f$  to  $f' : G' \rightarrow G'$  by

$$A' \mapsto A' \sigma \bar{B}' \sigma B' \quad B' \mapsto B' \sigma A' \sigma \bar{B}' \sigma B'$$

Then  $f' : G' \rightarrow G'$  is a train track map with EG strata  $H_2$  and  $H_3$  and associated laminations  $\Lambda_1$  and  $\Lambda_2$ . As in Example 4.1,  $\mathcal{F}(\Lambda_1)$  carries  $[F_m]$  and the conjugacy classes determined by the loops  $A$  and  $B$ ; and  $\mathcal{F}(\Lambda_2)$  carries  $[F_m]$  and the conjugacy classes determined by the loops  $A'$  and  $B'$ . If  $\mathcal{F}$  is a free factor system that carries both  $\Lambda_1$  and  $\Lambda_2$  then the component of  $\mathcal{F}$  that carries  $\Lambda_1$  and the component of  $\mathcal{F}$  that carries  $\Lambda_2$  both carry  $[F_m]$  and so must be equal. Thus  $\mathcal{F}$  is a single free factor that carries conjugacy classes that generate  $H_1(F_{m+4})$ . It follows that  $\mathcal{F}$  is not a proper free factor and so  $\mathcal{L}(\phi) = \{\Lambda_1, \Lambda_2\}$  fills even though neither  $\Lambda_1$  nor  $\Lambda_2$  fills.

The three items of Theorem 1.1 are proved separately: item (3) in Lemma 4.3; item (2) (assuming that (1) has been proved) in Lemma 4.4; and then item (1) in Corollary 4.21.

#### 4.1 Proof of Theorem 1.1 (3): Periodic vertices.

If the action of an outer automorphism on  $\mathcal{FS}(F_n)$  fixes a point then it permutes the vertices of the simplex whose interior contains that point and so has an iterate that fixes a vertex.

**Lemma 4.3.** *For all  $\phi \in \text{Out}(F_n)$ , the following are equivalent.*

- (a) *The action of some iterate of  $\phi$  on  $\mathcal{FS}(F_n)$  fixes a vertex.*
- (b) *The action of some iterate of  $\phi$  on  $\mathcal{FS}'(F_n)$  fixes a vertex.*
- (c)  *$\mathcal{L}(\phi)$  does not fill.*

*Proof.* It is obvious that (a) implies (b).

Assuming that (b) holds we will prove (c). There exist  $k \geq 1$  and a marked graph pair  $(G, H)$  such that  $\langle G, H \rangle^{\phi^k} = \langle G, H \rangle$ . Equivalently, there is a homotopy equivalence  $h : (G, H) \rightarrow (G, H)$  so that  $h : (G, H) \rightarrow (G, H)^{\phi^k}$  satisfies Definition 3.2. In particular,  $h : G \rightarrow G$  represents  $\phi^k$ . Note that for all  $m \geq 1$ , Definition 3.2 is satisfied by  $h : (G, H)^{\phi^{(m-1)k}} \rightarrow (G, H)^{\phi^{mk}}$  and hence also by  $h^m : (G, H) \rightarrow (G, H)^{\phi^{mk}}$ . Since the bijection that  $h$  induces on the natural edges of  $G \setminus H$  has finite order, we may choose  $m > 1$  so that the bijection induced by  $h^m$  on natural edges is the identity. Choosing such an  $m$  and replacing  $\phi$  by  $\phi^{mk}$  and  $h$  by  $h^m$  we may assume that  $h$  represents  $\phi$  and that  $h(E) = \mu E \nu$  for each edge  $E$  of  $G \setminus H$  and for paths  $\mu$  and  $\nu$ , dependent on  $E$ , that are either trivial or contained in  $H$ . It follows that  $\mathcal{L}(\phi)$  is carried by  $[H]$  and does not fill. (One way to see this is to note that for each circuit  $\sigma \subset G$  there is a uniform bound to the number of times



that an edge in  $G \setminus H$  is crossed by  $h_{\#}^i(\sigma)$ . This proves that if  $\sigma$  is weakly attracted to  $\Lambda \in \mathcal{L}(\phi)$  then  $\Lambda$  is carried by  $[H]$ . Since  $\sigma$  is arbitrary and every element of  $\mathcal{L}(\phi)$  weakly attracts at least one circuit,  $\mathcal{L}(\phi)$  is carried by  $[H]$ .)

To prove that (c) implies (a), suppose that  $\mathcal{L}(\phi)$  does not fill. After replacing  $\phi$  with an iterate we may assume that  $\phi$  is rotationless. Since  $\mathcal{L}(\phi)$  is  $\phi$ -invariant we may apply Theorem 4.28 of [FH11] to conclude that  $\phi$  is represented by a CT  $f : G \rightarrow G$  such that  $\mathcal{L}(\phi)$  is carried by a proper  $f$ -invariant subgraph  $G_i$  of  $G$ . The highest stratum  $H_N$  in the filtration is therefore a single edge  $E$  satisfying  $f(E) = uEv$  where  $u, v$  are paths in the  $f$ -invariant subgraph  $G_{N-1}$ . This proves that  $f : (G, G_{N-1}) \rightarrow (G, G_{N-1})^\phi$  satisfies Definition 3.2 and hence that  $\langle G, G_{N-1} \rangle$  is  $\phi$ -invariant.  $\square$

## 4.2 Proof of Theorem 1.1 (2): Nonloxodromic implies bounded orbits.

The next lemma gives a lamination criterion for verifying that an outer automorphism acts with bounded orbits on  $\mathcal{FS}(F_n)$ . This lemma immediately implies the “only if” direction of (1), and it reduces the proof of (2) to the “if” direction of (1) which will be proved later.

**Lemma 4.4.** *If  $\phi \in \text{Out}(F_n)$  and if each element of  $\mathcal{L}(\phi)$  is carried by a proper free factor system then the action of  $\phi$  on  $\mathcal{FS}(F_n)$  has bounded orbits.*

The idea of the proof is to show that a rotationless power of  $\phi$  has a topological representative having the structure of Example 4.2, and to use that structure to prove that  $\phi$  has bounded orbits on  $\mathcal{FS}(F_n)$ . The following technical lemma describes the structure that we use. The lemma will be proved after using it to prove Lemma 4.4.

**Lemma 4.5.** *Suppose that  $\phi \in \text{Out}(F_n)$  is rotationless, that  $\mathcal{L}(\phi)$  fills and that no single element of  $\mathcal{L}(\phi)$  fills. Then there is a homotopy equivalence  $f : G \rightarrow G$  representing  $\phi$  and  $f$ -invariant proper subgraphs  $K_1$  and  $K_2$  such that:*

- (1)  $G = K_1 \cup K_2$ .
- (2)  $K_1$  is a core subgraph whose frontier vertices are fixed by  $f$ .
- (3) Every component of  $K_2$  is non-contractible.
- (4) The core  $J_2$  of  $K_2$  is  $f$ -invariant. Furthermore, each non-fixed edge  $E \subset K_2 \setminus J_2$  can be oriented so that its terminal endpoint is in  $J_2$  and so that  $f(E) = Eu$  for some non-trivial circuit  $u \subset J_2$ .

**Proof of Lemma 4.4:** We may assume by Lemma 4.3 that  $\mathcal{L}(\phi)$  fills. After replacing  $\phi$  with an iterate if necessary, we may also assume that  $\phi$  is rotationless. Choose  $f : G \rightarrow G$ ,  $K_1$  and  $J_2 \subset K_2$  as in Lemma 4.5. Let  $J_3$  be the core of  $K_1 \cap J_2$ . It suffices to show that for all  $k \geq 1$  there is a path in  $\mathcal{FS}'(F_n)$  of length at most four between  $\langle G, J_3 \rangle$  and  $\langle G, J_3 \rangle^{f^k}$ .

Define  $f_1 : G \rightarrow G$  to agree with  $f$  on  $K_1$  and to be the identity on  $G \setminus K_1$ . Continuity of  $f_1$  follows from (2). The restriction of  $f_1$  to  $K_1$  is a homotopy equivalence by Lemma 6.0.6 of [BFH00] and so  $f_1$  is a homotopy equivalence. Define  $f_2 : G \rightarrow G$  to be the identity on  $K_1$  and to agree with  $f$  on  $G \setminus K_1$ . Then  $f_2$  is continuous for the same reason that  $f_1$  is and  $f = f_2 f_1$  which implies that  $f_2$  is a homotopy equivalence.

The length  $\leq 4$  path of vertices in  $\mathcal{FS}'(F_n)$

$$\langle G, J_3 \rangle \longrightarrow \langle G, K_1 \rangle = \langle G, K_1 \rangle^{f_1^k} \longleftarrow \langle G, J_3 \rangle^{f_1^k} \longrightarrow \langle G, J_2 \rangle^{f_1^k} = \langle G, J_2 \rangle^{f^k} \longleftarrow \langle G, J_3 \rangle^{f^k}$$

in which the vertices connected by an arrow are either equal or bound an edge in  $\mathcal{FS}'(F_n)$  is justified as follows.

The subgraphs  $K_1$ ,  $J_2$  and  $J_3$  are proper core subgraphs and hence natural subgraphs of  $G$ . Thus each of the seven vertices is well defined. Since each arrow is induced by an inclusion of natural subgraphs, the vertices that each connects are either equal or bound an edge in  $\mathcal{FS}'(F_n)$  by Lemma 3.5. By construction,  $f_1^k : (G, K_1) \rightarrow (G, K_1)^{f_1^k}$  satisfies Definition 3.2 so  $\langle G, K_1 \rangle = \langle G, K_1 \rangle^{f_1^k}$ . Lemma 4.5 (4) implies that  $f_2^k : (G, J_2)^{f_1^k} \rightarrow (G, J_2)^{f_2^k f_1^k}$  satisfies Definition 3.2 so  $\langle G, J_2 \rangle^{f_1^k} = \langle G, J_2 \rangle^{f_2^k f_1^k} = \langle G, J_2 \rangle^{f^k}$ .  $\square$

**Remark 4.6.** A version of the above ‘distance four’ argument is used in [HM13e] to correct an error in an early version of that paper; see the Remark between Steps 2 and 3 of the proof of Proposition 6.5 of [HM13e].

We will need the following fact in the proof of Lemma 4.5.

**Fact 4.7.** *In any CT, any height  $j$  indivisible Nielsen path  $\rho$  decomposes as an alternating concatenation of maximal subpaths in  $H_j$  and Nielsen subpaths of height  $< j$ . As a consequence, any one-edge subpath  $E$  of  $\rho$  of height  $i < j$  is contained in a height  $i$  Nielsen subpath of  $\rho$ .*

This fact follows from (NEG Nielsen Paths) in the case when  $H_j$  is NEG, and from [FH11] Lemma 4.24 when  $H_j$  is EG.

**Proof of Lemma 4.5:** Let  $\mathcal{L}^h$  be the set of  $\Lambda \in \mathcal{L}(\phi)$  for which there exists a CT representing  $\phi$  whose top stratum corresponds to  $\Lambda$ . Then  $\mathcal{L}^h \neq \emptyset$  because  $\mathcal{L}(\phi)$  fills, which implies that the highest stratum in any CT representing  $\phi$  is EG. Moreover,  $\mathcal{L}^h$  contains at least two elements because the smallest free factor system carrying any element of  $\mathcal{L}(\phi)$  is proper and so is realized by a proper filtration element in some CT representing  $\phi$ . Note also that each element of  $\mathcal{L}^h$  is topmost, meaning that it is not contained in any other element of  $\mathcal{L}(\phi)$ .

Choose  $\Lambda_1, \Lambda_2 \in \mathcal{L}^h$  and a CT  $f : G \rightarrow G$  representing  $\phi$  and satisfying the property that there exists a proper core filtration element  $G_t$  such that  $[G_t]$  is the free factor support of the set  $\mathcal{L}(\phi) - \{\Lambda_1, \Lambda_2\}$ . The highest stratum  $H_N$  must be an EG stratum corresponding to one of  $\Lambda_1, \Lambda_2$ . If exactly one of these two laminations, say  $\Lambda_2$ , is geometric then we choose  $f : G \rightarrow G$  so that  $\Lambda_2$  corresponds to  $H_N$ , which can be done by insisting that the free factor support of the set  $\mathcal{L}(\phi) - \{\Lambda_2\}$  is also realized by a core filtration element. For now these are the only properties of  $f$  we use. Later we add a further constraint to  $f$  in the case that both  $\Lambda_1$  and  $\Lambda_2$  are geometric. To fix notation, we assume that  $\Lambda_2$  corresponds to  $H_N$  and let  $H_r$  be the stratum corresponding to  $\Lambda_1$ . Since  $\Lambda_1 \not\subset \Lambda_2$  it follows that no edge of  $H_N$  is weakly attracted to  $\Lambda_1$  and hence no term in the complete splitting of an  $f_\#$ -iterate of an edge of  $H_N$  is weakly attracted to  $\Lambda_1$ . For any  $i$  such that  $r < i < N$  note that  $H_i$  is not an EG stratum.

We next analyze weak attraction properties of edges of height  $> r$  by proving that the following hold for each edge  $E$  with height  $> r$ .

- (i) If  $E$  is non-fixed and non-linear edge then  $E$  occurs as a term in the complete splitting of  $f_{\#}^k(E')$  for some edge  $E'$  of  $H_N$  and some  $k \geq 1$ .
- (ii)  $E$  is not weakly attracted to  $\Lambda_1$ .
- (iii) If a term  $\mu$  in the complete splitting of  $f_{\#}^k(E)$  crosses an edge of  $H_r$  then  $\mu$  is a Nielsen path or an exceptional path.

If  $E$  belongs to a zero stratum then (i) follows from (Zero Strata) and the fact that  $H_N$  is the only EG stratum with height greater than  $r$ . If  $E$  is a non-fixed non-linear irreducible edge then  $E$  is crossed by  $\Lambda_2$  because  $\mathcal{L}(\phi)$  fills. It follows that  $E$  is contained in a term  $\nu$  of the complete splitting of  $f_{\#}^k(E')$  for some edge  $E'$  of  $H_N$  and some  $k \geq 1$ . Item (i) then follows from Fact 4.7 which implies that  $\nu$  is a single edge. Item (ii) is obvious if  $E_i$  is fixed or linear; in the remaining cases (ii) follows from (i) and the fact that no edge  $E'$  of  $H_N$  is weakly attracted to  $\Lambda_1$ . To prove (iii) we must show that  $\mu$  is neither a subpath of a zero stratum nor a single edge. The former is obvious and the latter follows from (ii).

We now verify conclusions (1)–(4) of Lemma 4.5 under a special assumption. Let  $K_1 = G_r$  and note that  $K_1$  is an  $f$ -invariant core subgraph. By [FH11, Remark 4.9] and (Zero Strata), each point in the frontier of  $K_1$  is a principal vertex and is hence fixed by  $f$ . Item (2) is therefore satisfied. Next we formulate:

*Special Assumption:*  $G \setminus H_r^z$  is  $f$ -invariant.

Setting  $K_2 = G \setminus H_r^z$ , item (1) is obvious. Since  $K_2$  is  $f$ -invariant and contains  $H_N$ , it follows that  $K_2$ , and hence  $J_2 = \text{core}(K_2)$ , contains the realization of  $\Lambda_2$  in  $G$ . Lemma 4.15 of [FH11] implies that the components of  $G_r \setminus H_r^z$  are non-contractible. Since the remaining edges  $K_2 \setminus (G_r \setminus H_r^z)$  all have height  $> r$  and so are crossed by  $\Lambda_2$ , every component of  $K_2$  is non-contractible and item (3) is satisfied. To prove (4), let  $G_{r'}$  be the highest core filtration element that is properly contained in  $G_r$ . Then  $G_r$  is built up from  $G_{r'}$  by first adding NEG edges, if any, and then adding the zero strata, if any, that are enveloped by  $H_r$  and then adding  $H_r$ . If  $E$  is one of the NEG edges then (Periodic edges) implies that  $E$  is not fixed and so can be oriented so that  $f(E) = Eu$  for some circuit  $E$  that is necessarily contained in  $G_{r'}$ . The core graph  $J_2$  contains  $G_{r'}$  and all edges crossed by  $\Lambda_2$  and so is obtained from  $K_2$  by removing a (possibly empty) set of the NEG edges with height between  $r'$  and  $r$ . To verify (4) it remains only to show that if  $E$  has height  $> r$ , and so in particular  $E \subset J_2$ , then  $f(E) \subset J_2$ ; that will prove that  $J_2$  is  $f$ -invariant. If  $E$  is fixed this is obvious. If  $E$  is linear then  $f(E) = Eu$  where  $u$  is a circuit that is contained in  $K_2$  and hence in  $J_2$ . Otherwise, by (i) it follows that  $f(E)$  is a term of the complete splitting of  $f_{\#}^{k+1}(E')$  for some edge  $E' \subset H_N$  and some  $k \geq 1$ , so  $f(E)$  is a subpath of  $\Lambda_2$  and hence is contained in  $J_2$ .

It remains to justify the *Special Assumption*. We divide into three cases, in two of which we indeed verify that the *Special Assumption* is true, completing the proof in those cases. The third requires careful handling, in that the *Special Assumption* can fail.

**Case 1:  $\Lambda_1$  is non-geometric and there are no Nielsen paths of height  $r$ .**

By Fact 4.7, no Nielsen path (and hence no exceptional path) crosses an edge of  $H_r$ . Applying (iii), it therefore follows that if  $E$  is an edge with height  $> r$  then  $f(E)$  does not cross

an edge of  $H_r$ . If  $f(E)$  intersects a zero stratum enveloped by  $H_r$  then  $f(E)$  is contained in that zero stratum because it does not cross any edges of  $H_r$ . It follows that  $E$  must be contained in a zero stratum enveloped by  $H_N$ ; but then the  $f$ -image of an edge in  $H_N$  would cross an edge of  $H_r$ . This contradiction implies that  $f(E) \subset G \setminus H_r^z$ . We have already seen that  $G_r \setminus H_r^z$  is  $f$ -invariant, and so  $G \setminus H_r^z$  is  $f$ -invariant.

**Case 2:  $\Lambda_1$  and  $\Lambda_2$  are geometric.** For  $i = 1, 2$ , choose  $f_i : G^i \rightarrow G^i$  representing  $\phi$  in which  $\Lambda_i$  corresponds to the highest stratum  $H_{N_i}^i$ . Let  $\rho_i$  be the unique indivisible Nielsen path in  $G^i$  of height  $N_i$ , a closed path representing a conjugacy class in  $F_n$  that we denote  $c_i = [\rho_i]$ . Also let  $[\langle \rho_i \rangle]$  denote the conjugacy class of the infinite cyclic subgroup of  $F_n$  generated by  $\rho_i$ . We have:

- (a) ([HM13c], Section 1.1, “Remark: The case of a top stratum”)  $[\langle \rho_i \rangle]$  is a rank 1 element of the subgroup system  $\mathcal{A}_{\text{na}}(\Lambda_i)$  and  $\mathcal{A}_{\text{na}}(\Lambda_i) - \{[\langle \rho_i \rangle]\}$  is carried by  $[G_{N_i} - 1]$ .
- (b) ([HM13b], Proposition 2.18 (2)) The conjugacy class  $c_i$  is an element of a finite  $\phi$ -invariant set  $\mathcal{C}_i$  of conjugacy classes such that  $\mathcal{F}_{\text{supp}}(\mathcal{C}_i) = \mathcal{F}_{\text{supp}}(\Lambda_i)$ .

Each individual element of  $\mathcal{L}(\phi)$  and of  $\mathcal{C}_1 \cup \mathcal{C}_2$  is  $\phi$ -invariant by Lemma 3.30 of [FH11]. The following set is therefore  $\phi$ -invariant:

$$L = (\mathcal{L}(\phi) \setminus \{\Lambda_1, \Lambda_2\}) \cup (\mathcal{C}_1 \setminus c_1) \cup (\mathcal{C}_2 \setminus c_2)$$

Since  $L$  contains  $\mathcal{L}(\phi) - \{\Lambda_1, \Lambda_2\}$ , the free factor support of  $L$  contains  $[G_t]$ , and so we may impose an additional constraint on the CT  $f: G \rightarrow G$  by requiring that there is a core filtration element  $G_s$  containing  $G_t$  such that  $[G_s]$  equals the free factor support of  $L$ . Note that  $G_r$  contains  $G_s$  because  $[G_r]$  carries  $\mathcal{L}(\phi) \setminus \{\Lambda_2\}$  which contains  $L$ . The proper free factor system  $[G_{N_1-1}^1]$  carries  $(\mathcal{L}(\phi) \setminus \Lambda_1) \cup (\mathcal{C}_1 \setminus c_1) \cup \mathcal{C}_2$  and the proper free factor system  $[G_{N_2-1}^2]$  carries  $(\mathcal{L}(\phi) \setminus \Lambda_2) \cup \mathcal{C}_1 \cup (\mathcal{C}_2 \setminus c_2)$ . It follows that  $[G_s] \sqsubset [G_{N_1-1}^1] \wedge [G_{N_2-1}^2]$  and hence neither  $c_1$  nor  $c_2$  is carried by  $G_s$ .

Let  $\rho$  be the unique indivisible Nielsen path in  $G$  of height  $r$  and let  $x$  be its basepoint. The elements of  $\mathcal{C}_1 - c_1$  are carried by  $G_s$ . Since  $G_{r-1}$  does not carry  $\Lambda_1$ , but  $G_r$  does,  $c_1$  is carried by  $G_r$  but not by  $G_{r-1}$ . Since  $c_1$  is carried by  $\mathcal{A}_{\text{na}}(\Lambda_1)$ , Proposition 2.18 (4) of [HM13b] implies that  $c_1 = [\rho]$  up to a change of orientation. Lemma 4.24 of [FH11] implies that  $H_r^z = H_r$  and  $H_N^z = H_N$ . We proved (ii) earlier saying that no edge with height  $> r$  is weakly attracted to  $\Lambda_1$ , and the same is clearly true of edges of height  $< r$ . It follows that there does not exist any closed path in  $G \setminus H_r$  whose base point equals the base point  $w$  of  $\rho_1$ , for if such a closed path existed then together with  $\rho$  it would generate a rank 2 subgroup of  $F_n$  supported by  $\mathcal{A}_{\text{na}}(\Lambda_1)$  and containing  $\rho$ , contradicting item (a) and the fact that the subgroup system  $\mathcal{A}_{\text{na}}(\Lambda_1)$  is malnormal [HM13c, Proposition 1.4]. By (iii), each non-trivial maximal subpath of  $\Lambda_2$  contained in  $H_r$  is a copy of  $\rho$  or  $\rho^{-1}$  and so begins and ends at  $w$ ; if any such subpaths exist then their complementary subpaths in  $\Lambda_2$  would begin and end at  $w$  which is a contradiction. Thus  $\Lambda_2 \subset G \setminus H_r$  and the point  $w$  is not crossed by  $\Lambda_2$ . Since  $\Lambda_2$  crosses every edge with height  $> r$ , no such edge is incident to  $w$ . It therefore follows from (iii) that  $f(E) \subset G - H_r$  for each edge  $E$  with height  $> r$ . This completes the proof that  $G \setminus H_r^z$  is  $f$ -invariant.

**Case 3:  $\Lambda_1$  is non-geometric and there is a Nielsen path of height  $r$ .**

In this case  $H_r^z = H_r$  by Lemma 4.24 of [FH11], and we set  $K'_2 = G \setminus H_r$ . But  $K'_2$  is not necessarily  $f$ -invariant. The height  $r$  Nielsen path  $\rho_r$  is unique up to orientation, it has distinct endpoints, and it may be oriented with initial vertex  $p$  and terminal vertex  $q \notin G_{r-1}$  (see Fact 1.42 of [HM13b]). In terms of  $\rho_r$  we shall see exactly how invariance of  $K'_2$  can fail, and using the mode of failure as a guideline we shall then modify  $f: G \rightarrow G$ .

Any Nielsen path (and hence any exceptional path) that crosses an edge in  $H_r$  decomposes as a concatenation of subpaths that are either contained in  $K'_2$  or equal to  $\rho_r$  or  $\rho_r^{-1}$ —this follows by applying Fact 4.7 to each of the fixed edges and indivisible Nielsen paths into which the given Nielsen path decomposes. Combining this with (iii) it follows that if  $E$  is an edge in  $K'_2$  then  $f(E)$  decomposes as a concatenation of subpaths  $f(E) = \sigma_1 \dots \sigma_m$  where each  $\sigma_j$  is either contained in  $K'_2$  or equal to  $\rho_r$  or  $\rho_r^{-1}$ .

Let  $\mathcal{D}$  be the set of directions in  $G \setminus G_r$  that are based at the terminal point  $q$  of  $\rho_r$ . If  $\mathcal{D} = \emptyset$  then for each edge  $E \subset K'_2$  the path  $f(E)$ , which has endpoints in  $K'_2$ , must be contained in  $K'_2$ . The subgraph  $K'_2$  is therefore  $f$ -invariant, we set  $K_2 = K'_2$ , and we are done.

Assuming that  $\mathcal{D} \neq \emptyset$ , we modify  $f: G \rightarrow G$  as follows. Define a new graph  $G'$  by detaching the edges associated to  $\mathcal{D}$  from  $q$ , reattaching them to a new vertex  $q'$  and adding a new oriented edge  $X'$  with initial endpoint  $p$  and terminal endpoint  $q'$ . We view  $K_1 = G_r$  and  $H_r \subset G_r$  as subgraphs of both  $G$  and  $G'$ . Folding  $X'$  with  $\rho_r$  determines a homotopy equivalence  $h: G' \rightarrow G$  that we use to mark  $G'$ . Letting  $Z = K'_2 = G \setminus H_r$  and  $Z' = G' \setminus (H_r \cup X')$ , the restriction  $h|_{Z'}: Z' \rightarrow Z$  is a homeomorphism. We may identify edges and edge paths in  $Z'$  with edges and edge paths in  $Z$  via the map  $h$ , writing  $E' \leftrightarrow E$  for corresponding edges  $E' \subset Z'$ ,  $E \subset Z$ , and  $\tau' \leftrightarrow \tau$  for corresponding edge paths.

The homotopy equivalence  $f: G \rightarrow G$  induces a homotopy equivalence  $f': G' \rightarrow G'$  representing  $\phi$  that agrees with  $f$  on  $G_r$ , that fixes  $X'$ , and that takes each edge  $E' \subset Z'$  to  $f'(E') = \tau_1 \dots \tau_m$  where  $f(E) = \sigma_1 \dots \sigma_m$  is as above and where  $\tau_j = \sigma'_j \subset Z'$  if  $\sigma_j \subset Z$  and  $\tau_j = X'$  or  $X'^{-1}$  if  $\sigma_j = \rho_r$  or  $\rho_r^{-1}$  respectively. This proves that  $K_2 = Z' \cup X'$  is  $f'$ -invariant. We leave it to the reader to verify the remaining conclusions of Lemma 4.5 as they were proved earlier under the *Special Assumption*.

This completes the proof of Lemma 4.5, which completes the proof of Theorem 1.1 (2) modulo the proof of (1).  $\square$

### 4.3 Lemmas on filling laminations

Preparatory to the proof that existence of a filling lamination is sufficient for an outer automorphism to act loxodromically on  $\mathcal{FS}(F_n)$ , in this section we prove some facts about filling laminations.

Every free factor system  $\mathcal{F}$  is realized as a core subgraph  $H$  of some marked graph  $G$ . If  $k$  is the minimum number of natural edges of  $G$  contained in  $G - H$  over all choices of such a pair  $H \subset G$ , then we say that  $\mathcal{F}$  is a *co-edge  $k$  free factor system* and that  $k$  is the *co-edge number* for  $\mathcal{F}$ .

**Lemma 4.8.** (1) If  $\mathcal{F} = \{[F_1], \dots, [F_p]\}$  and if  $r_i$  is the rank of  $F_i$  then the co-edge number of  $\mathcal{F}$  is  $k = (n - \sum r_i) + (p - 1) = (n - 1) - \sum_{i=1}^p (r_i - 1)$ .

- (2) Suppose that  $\mathcal{F}_i$ ,  $i = 1, 2$ , is a free factor system with co-edge number  $k_i$  and that  $\mathcal{F}_1 \sqsubset \mathcal{F}_2$ . Then  $k_2 \leq k_1$  with equality if and only if  $\mathcal{F}_1$  is obtained from  $\mathcal{F}_2$  by removing  $\leq k_2$  rank one components from  $\mathcal{F}_2$ .

**Remark 4.9.** We note the following consequences of Lemma 4.8. First, given a free factor system  $\mathcal{F}$ , the extension  $\mathcal{F} \sqsubset \{[F_n]\}$  is a one edge extension (see Section 2) if and only if  $\mathcal{F}$  has co-edge number 1. More generally, an extension of free factor systems  $\mathcal{F} = \{[F_1], \dots, [F_p]\} \sqsubset \{[F'_1], \dots, [F'_{p'}]\} = \mathcal{F}'$  is a one-edge extension if and only if one of the following two alternatives holds: each component of  $\mathcal{F}'$  contains some component of  $\mathcal{F}$  and  $\sum_{i=1}^{p'} (\text{rank } F'_i - 1) - \sum_{j=1}^p (\text{rank}(F_j) - 1) = 1$ ; or  $\mathcal{F}'$  is the union of  $\mathcal{F}$  and a rank 1 component.

*Proof.* There is no loss in restricting attention to pairs  $H \subset G$  in which each component of  $H$  is a rose and to simplicial structures on  $G$  in which all vertices have valence at least three. In the context of (1),  $H$  has  $p$  vertices and  $\sum r_i$  edges. If  $G$  has  $q$  additional vertices then the obvious Euler characteristic calculation shows that the number of natural edges in  $G - H$  is  $(n - \sum r_i) + (p - 1) + q$ , which is minimized by choosing  $q = 0$ . This proves (1).

For (2) define  $\mathcal{F}'_2$  by removing components from  $\mathcal{F}_2$  that do not contain any element carried by  $\mathcal{F}_1$  and let  $k'_2$  be the co-edge number of  $\mathcal{F}'_2$ . Then  $\mathcal{F}_1 \sqsubset \mathcal{F}'_2 \sqsubset \mathcal{F}_2$  and the second formula in (1) implies that  $k_2 \leq k'_2$  with equality if and only if each removed component has rank one. Let  $n_1$  and  $n'_2$  be the sum of the ranks of the components of  $\mathcal{F}_1$  and  $\mathcal{F}'_2$  respectively and let  $p_1$  and  $p'_2$  be the number of components of  $\mathcal{F}_1$  and  $\mathcal{F}'_2$  respectively. Then  $p'_2 \leq p_1$  and  $n'_2 \geq n_1$  so the first formula in (1) implies that  $k'_2 \leq k_1$  with equality if and only if  $p'_2 = p_1$  and  $n'_2 = n_1$ . Thus  $k_2 \leq k_1$  with equality if and only if each removed component of  $\mathcal{F}_2$  has rank one and  $\mathcal{F}_1 = \mathcal{F}'_2$ . Since  $k_2 \geq p_2 - 1$ , at most  $k_2$  such components can be removed. This completes the proof of (2).  $\square$

**Lemma 4.10.** Suppose that  $\phi$  is rotationless, that  $\Lambda \in \mathcal{L}(\phi)$  fills and that  $\mathcal{A} = \mathcal{A}_{na}(\Lambda)$  is its non-attracting subgroup system.

- (1) If  $\Lambda$  is non-geometric then  $\mathcal{A}$  is a co-edge  $\geq 2$  free factor system.
- (2) If  $\Lambda$  is geometric then there is a rank one component of  $\mathcal{A}$  whose complement in  $\mathcal{A}$  is a co-edge  $\geq 2$  free factor system.

*Proof.* Since  $\Lambda$  fills it is associated to the top stratum in any CT representing  $\phi$ . The Remark preceding Remark 1.3 of [HM13c] implies that  $\mathcal{A}$  is a  $\phi$ -invariant free factor system if  $\Lambda$  is non-geometric and is obtained from a  $\phi$ -invariant free factor system by adding a rank one component if  $\Lambda$  is geometric. The lemma therefore follows from Corollary 3.32 of [BFH00] which implies that  $\phi$  does not preserve any co-edge one free factor system.  $\square$

**Lemma 4.11.** Suppose that  $\phi$  is rotationless, that  $\Lambda \in \mathcal{L}(\phi)$  fills and that  $\mathcal{A} = \mathcal{A}_{na}(\Lambda)$  is its non-attracting subgroup system.

- (1) If  $\mathcal{F}$  is a co-edge one free factor system then  $\mathcal{F}$  carries a conjugacy class that is not carried by  $\mathcal{A}$ .
- (2) If  $\mathcal{F}$  is a co-edge two free factor system then one of the following holds.



- (a)  $\mathcal{F}$  carries a conjugacy class that is not carried by  $\mathcal{A}$ .
- (b)  $\mathcal{F}$  is obtained from  $\mathcal{A}$  by removing  $\leq 3$  rank one components.

*Proof.* If  $\Lambda$  is non-geometric then  $\mathcal{A}$  is a co-edge  $\geq 2$  free factor system by Lemma 4.10(1). The free factor system  $\mathcal{F} \wedge \mathcal{A}$  is contained in  $\mathcal{F}$  and carries all the conjugacy classes of  $\mathcal{F}$  that are carried by  $\mathcal{A}$ . If  $\mathcal{F} \wedge \mathcal{A} \neq \mathcal{F}$  then  $\mathcal{F}$  carries a conjugacy class that is not carried by  $\mathcal{A}$  and we are done. If  $\mathcal{F} \wedge \mathcal{A} = \mathcal{F}$  then  $\mathcal{F} \sqsubset \mathcal{A}$ . Since  $\mathcal{F}$  has co-edge number  $\leq 2$  and  $\mathcal{A}$  has co-edge number  $\geq 2$ , Lemma 4.8(2) implies that both  $\mathcal{F}$  and  $\mathcal{A}$  have co-edge number two and that  $\mathcal{F}$  is obtained from  $\mathcal{A}$  by removing  $\leq 2$  rank one components.

If  $\Lambda$  is geometric then by Lemma 4.10(2)) there is a rank one component  $[\langle \rho \rangle]$  of  $\mathcal{A}$  whose complement  $\mathcal{A}'$  in  $\mathcal{A}$  is a co-edge  $\geq 2$  free factor system. If  $[F]$  is a component of  $\mathcal{F}$  and  $[F] \wedge \mathcal{A}' \sqsubset [F]$  is a proper inclusion, then there are many conjugacy classes in  $[F]$  that are not carried by  $\mathcal{A}'$  and we can choose one that is not contained in  $[\langle \rho \rangle]$  and hence not carried by  $\mathcal{A}$ . We are therefore reduced to the case that  $\mathcal{F} \sqsubset \mathcal{A}'$ . As above, it follows that  $\mathcal{F}$  and  $\mathcal{A}'$  have co-edge number two and that  $\mathcal{F}$  is obtained from  $\mathcal{A}'$  by removing  $\leq 2$  rank one component. Thus  $\mathcal{F}$  is obtained from  $\mathcal{A}$  by removing  $\leq 3$  rank one components and we are done.  $\square$

**Example 4.12.** Let  $h : S \rightarrow S$  be a pseudo-Anosov homeomorphism of the orientable genus zero surface  $S$  with four boundary components and let  $\phi \in \text{Out}(F_3)$  be the outer automorphism of  $F_3$  determined by  $h$  and an identification of  $F_3$  with the fundamental group of  $S$ . Then  $\mathcal{L}(\phi)$  has a single filling element  $\Lambda$  and  $\mathcal{A}_{\text{na}}\Lambda$  has four rank one components, one for each component of  $\partial S$ . Any one, two, or three of these components form a co-edge two free factor system. Similar examples can be made in which  $\phi$  has a geometric stratum but is not itself geometric.

#### 4.4 Proof of Theorem 1.1 (1): Characterization of loxodromics

As explained at the beginning of Section 4.2, to prove (1) all that remains is to prove that if  $\phi \in \mathcal{L}(\phi)$  has an attracting lamination  $\Lambda_\phi^+ \in \mathcal{L}(\phi)$  which fills  $F_n$  then  $\phi$  acts loxodromically on  $\mathcal{FS}(F_n)$ ; see Corollary 4.21 below. The method of proof is to construct a map  $W : \mathcal{FS}(F_n) \mapsto \mathbb{Z}$  (Definition 4.15) which is equivariant with respect to the cyclic group  $\langle \phi^i \rangle$  acting on  $\mathcal{FS}(F_n)$  and on  $\mathbb{Z}$  where  $\phi^i \cdot j = i + j$  (Remark 4.19) and which is Lipschitz (Lemma 4.20).

The method of proof we use applies as well to show the result from [BF11], Theorem 9.3, saying that if  $\phi \in \text{Out}(F_n)$  is fully irreducible then  $\phi$  acts loxodromically on the free factor complex; see Remark 4.22.

The following lemma is used below, in Lemmas 4.14 and 4.23, in order to apply Proposition 3.1 of [HM13d].

**Lemma 4.13.** *Suppose that  $\Lambda_\phi^+ \in \mathcal{L}(\phi)$  and  $\Lambda_\psi^+ \in \mathcal{L}(\psi)$  are filling laminations with generic leaves  $\gamma_\phi$  and  $\gamma_\psi$  respectively. Assume that  $\Lambda_\phi^+ \neq \Lambda_\psi^+$ . Then there is a proper free factor system that carries  $\Lambda_\phi^+ \cap \Lambda_\psi^+$  and does not carry an end of either  $\gamma_\phi$  or  $\gamma_\psi$ .*

*Proof.* Assuming without loss that  $\Lambda_\phi^+ \not\subset \Lambda_\psi^+$ , no leaf of  $\Lambda_\phi^+ \cap \Lambda_\psi^+$  has closure equal to  $\Lambda_\phi^+$ . The existence of a proper free factor system carrying  $\Lambda_\phi^+ \cap \Lambda_\psi^+$  therefore follows from

Lemma 3.1.15 of [BFH00]. Since  $\gamma_\phi$  and  $\gamma_\psi$  are birecurrent and filling their ends are not carried by any proper free factor system.  $\square$

Suppose that  $\phi, \phi^{-1} \in \text{Out}(F_n)$  are rotationless, that  $\Lambda^\pm$  is a lamination pair for  $\phi$  and that the conjugacy class  $c$  is not carried by  $\mathcal{A}_{\text{na}}(\Lambda^\pm)$ . As  $i \rightarrow \infty$ ,  $\phi^i(c)$  contains longer and longer subpaths in common with  $\Lambda^+$  and does not contain any very long subpaths of  $\Lambda^-$ . Symmetrically,  $\phi^{-i}(c)$  contains longer and longer subpaths in common with  $\Lambda^-$  and does not contain any very long subpaths of  $\Lambda^+$ . In the middle, there is an interval of integers  $w$  such that  $\phi^w(c)$  has neither very long subpaths of  $\Lambda^+$  nor very long subpaths of  $\Lambda^-$ . The following lemma makes this precise in our current context. See [AK11] for a similar definition.

**Lemma 4.14.** *Suppose that  $\phi, \phi^{-1} \in \text{Out}(F_n)$  are rotationless, that  $\Lambda^\pm$  is a lamination pair for  $\phi$  that fills and that  $\mathcal{A}_{\text{na}}\Lambda^\pm$  is its non-attracting subgroup system. Let  $\gamma^+, \gamma^-$  be generic leaves of  $\Lambda^+, \Lambda^-$  respectively. Then there exist neighborhoods  $U^-, U^+ \subset \mathcal{B}$  of  $\gamma^-$  and  $\gamma^+$  respectively, and there is a positive integer  $M$ , such that the following properties hold:*

- (1) *For each conjugacy class  $c$  that is not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  there exists a unique smallest  $w(c) \in \mathbb{Z}$  such that  $\phi^i(c) \in U^-$  for all  $i \leq -w(c)$ . Moreover,  $\phi^i(c) \in U^+$  for all  $i \geq -w(c) + M$ .*
- (2) *We have  $w(\phi^m(c)) = w(c) + m$  for any  $c$  as in (1).*
- (3) *If  $c_1$  and  $c_2$  are conjugacy classes that are not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  and that are both carried by some proper free factor system then  $|w(c_1) - w(c_2)| \leq M$ .*

*Proof.* Lemma 4.13 (applied with  $\psi = \phi^{-1}$ ) implies that  $B_1 = \{\gamma^+\}$  and  $B_2 = \{\gamma^-\}$  satisfy the hypotheses of Proposition 3.1 of [HM13d]. The conclusion of that proposition is the existence of neighborhoods  $U^+, U^- \subset \mathcal{B}$  of  $\gamma^+, \gamma^-$  respectively so that no proper free factor system carries both a conjugacy class carried by  $U^+$  and a conjugacy class carried by  $U^-$ . There is no loss (see Definition 3.1.5 of [BFH00]) in assuming that  $U^+$  and  $U^-$  are attracting neighborhoods, with respect to some iterate  $\phi^m$ , for  $\Lambda^+$  and  $\Lambda^-$  respectively. In particular,  $\{\phi^{-km}(U^-) : k \geq 0\}$  is a neighborhood basis in  $\mathcal{B}$  for  $\gamma^-$  (and hence a neighborhood basis for  $\Lambda^-$ ).

If  $c$  is not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  then  $\phi^{-j}(c) \in U^-$  for all sufficiently large  $j$  by Theorem F of [HM13c]. Since  $\gamma^-$  is not periodic, there exists a neighborhood of  $\gamma^-$  that does not carry  $c$  and so there exists  $k$  so that  $c \notin \phi^{-km}(U^-)$  or equivalently  $\phi^{km}(c) \notin U^-$ . This proves that  $w(c)$  is well defined and that the first part of (1) holds, and (2) evidently holds as well. By Theorem H of [HM13c] there exists  $M$  satisfying the rest of (1).

Suppose that  $c_1$  and  $c_2$  are conjugacy classes that are not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  and that are both carried by a proper free factor system  $\mathcal{F}$ . If  $w(c_1) - M > w(c_2)$  then the free factor system  $\phi^{-(w(c_1)-M)}(\mathcal{F})$  carries  $\phi^{-w(c_1)+M}(c_1) \in U^+$  and  $\phi^{-(w(c_1)-M)}(c_2) \in U^-$  which contradicts the above conclusion of Proposition 3.1 of [HM13d]. Thus  $w(c_1) - M \leq w(c_2)$ . The symmetric argument shows that  $w(c_2) - M \leq w(c_1)$  so (3) is satisfied.  $\square$

**Definition 4.15.** For the rest of the section we fix a rotationless  $\phi \in \text{Out}(F_n)$ , a lamination  $\Lambda^+ \in \mathcal{L}(\phi)$  that fills with dual lamination  $\Lambda^- \in \mathcal{L}(\phi)$ , and neighborhoods  $U^+, U^-$  of generic

leaves of  $\Lambda^+, \Lambda^-$  respectively, so that the conclusions of Lemma 4.14 hold, and in particular  $w(c)$  is defined as in (1).

Suppose that  $\mathcal{F}$  is a free factor system that carries at least one conjugacy class that is not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$ . Define  $W(\mathcal{F})$  to be the minimum value of  $w(c)$  as  $c$  varies over all conjugacy classes that are carried by  $\mathcal{F}$  and that are not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$ . Lemma 4.14 (3) guarantees that  $W(\mathcal{F})$  is well defined. If  $S$  is a one edge free splitting then its elliptic subgroups determine a co-edge one free factor system  $\mathcal{F}(S)$  and so  $W(S) = W(\mathcal{F}(S))$  is defined, because Lemma 4.11 (1) guarantees existence of a conjugacy class carried by  $\mathcal{F}(S)$  but not by  $\mathcal{A}_{\text{na}}(\Lambda^\pm)$ .

**Lemma 4.16.** *If  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are proper free factor systems and if there exists a conjugacy class  $c$  that is carried by  $\mathcal{F}_1$  and  $\mathcal{F}_2$  but not by  $\mathcal{A}_{\text{na}}\Lambda^\pm$ , then  $|W(\mathcal{F}_1) - W(\mathcal{F}_2)| \leq 2M$ .*

*Proof.*  $|W(\mathcal{F}_1) - W(\mathcal{F}_2)| \leq |W(\mathcal{F}_1) - W(c)| + |W(c) - W(\mathcal{F}_2)| \leq M + M$  by Lemma 4.14 (3).  $\square$

**Remark 4.17.** If  $w(c) \geq M$  then  $c \in U^+$ , whereas if  $w(c) \leq 0$  then  $c \in U_-$  (by Lemma 4.14 (1)). Thus  $W(\mathcal{F}) \geq M$  implies that every conjugacy class in  $\mathcal{F}$  that is not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  is carried by  $U^+$ , whereas  $W(\mathcal{F}) \leq -M$  implies that every conjugacy class of in  $\mathcal{F}$  that is not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  is carried by  $U^-$ .

**Remark 4.18.** Since  $\mathcal{A}_{\text{na}}\Lambda^\pm$  is  $\phi$ -invariant and  $c$  is carried by  $\mathcal{F}$  if and only if  $\phi^m(c)$  is carried by  $\phi^m(\mathcal{F})$ , applying Lemma 4.14 (2) we have  $W(\phi^m(\mathcal{F})) = W(\mathcal{F}) + m$  for all  $m$ .

**Remark 4.19.** Recall that  $[a]$  acts elliptically on a free splitting  $S$  if and only if  $\phi^{-1}[a]$  acts elliptically on  $S^\phi$ . Thus  $\mathcal{F}(S^\phi) = \phi^{-1}\mathcal{F}(S)$ . By the previous remark,  $W(S^{\phi^m}) = W(S^\phi) - m$  for all  $m$ .

**Lemma 4.20.** *If  $S_1$  and  $S_2$  are one-edge free splittings that bound an edge in  $\mathcal{FS}(F_n)$  then  $|W(S_1) - W(S_2)| \leq 8M$ .*

*Proof.* By Lemma 3.5 there is a co-edge two natural marked graph pair  $(G, H)$  with the edges of  $G \setminus H$  labelled  $E_1, E_2$  so that  $S_1 = \langle G, H \cup E_1 \rangle$  and  $S_2 = \langle G, H \cup E_2 \rangle$ . Recalling that  $n = \text{rank}(F_n) \geq 3$ , it follows that  $H$  is a nonempty subgraph of  $G$ ; recall also that each of its components is noncontractible. After collapsing a maximal subforest in  $H$ , we may assume that each component of  $H$  is a rose.

If  $[H]$  carries a conjugacy class that is not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  then Lemma 4.16, applied with  $\mathcal{F}_i = [H \cup E_i]$ , implies that  $|W(S_1) - W(S_2)| \leq 2M$ . We may therefore assume that every element of  $[H]$  is carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$ . By Lemma 4.11 (2), there exist  $x_1, x_2, x_3 \in F_n$  such that any conjugacy class carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$  but not by  $[H]$  is contained in the set  $\{[x_i^k] \mid 1 \leq i \leq 3, k \in \mathbb{Z}\}$ .

There are four cases to consider. The first three use the following method for bounding  $|W(S_1) - W(S_2)|$ .

Suppose that  $\sigma_0$  is a closed path in  $H$  and that for  $i = 1, 2$ ,  $\sigma_i$  is a closed path in  $H \cup E_i$  that is not contained in  $H$ . Suppose further that all three paths have a common basepoint  $v$  and that the elements  $a, b, c \in F_n$  determined by  $\sigma_0, \sigma_1$  and  $\sigma_2$  respectively (under an identification of  $\pi_1(G, v)$  with  $F_n$ ) are part of a free basis for  $F_n$ , generating a rank 3 free factor  $\langle a, b, c \rangle < F_n$ . Choose  $m > 0$  so that each of  $[ab^m], [ac^m], [bc^m]$  and  $[cb^m]$  is not

contained in  $\{[x_i^k]; 1 \leq i \leq 3; k \in \mathbb{Z}\}$  and hence not carried by  $\mathcal{A}_{\text{na}}\Lambda^\pm$ . Consider the rank two free factors

$$\underbrace{\langle a, b \rangle}_{X_1} \quad \underbrace{\langle ab^m, bc^m \rangle}_{X_2} \quad \underbrace{\langle b, c \rangle}_{X_3} \quad \underbrace{\langle ac^m, cb^m \rangle}_{X_4} \quad \underbrace{\langle a, c \rangle}_{X_5}$$

The elements

$$ab^m \quad bc^m \quad cb^m \quad ac^m$$

are contained in

$$X_1 \cap X_2 \quad X_2 \cap X_3 \quad X_3 \cap X_4 \quad X_4 \cap X_5$$

respectively. Since  $[\langle a, b \rangle] \subset [H \cup E_1]$  and  $[\langle a, c \rangle] \subset [H \cup E_2]$ , four applications of Lemma 4.16 imply that  $|W(S_1) - W(S_2)| \leq 8M$ .

We now turn to the case analysis.

If  $H$  is connected then  $G$  is a rose and we let  $\sigma_0$  be one of the loops in  $H$  and  $\sigma_i = E_i$  for  $i = 1, 2$ .

If  $H$  has three components, label them  $H_0, H_1$  and  $H_2$  where  $H_0$  contains the initial vertices of both  $E_1$  and  $E_2$  and  $H_i$  contains the terminal vertex of  $E_i$  for  $i = 1, 2$ . For  $j = 0, 1, 2$  let  $\tau_j$  be a loop based at the unique vertex of  $H_j$ . Let  $\sigma_0 = \tau_0$  and for  $i = 1, 2$  let  $\sigma_i = E_i \tau_i \overline{E_i}$ .

If  $H$  has two components  $H_0$  and  $H_1$  and one of  $E_1$  or  $E_2$  is a loop then we may assume that  $E_1$  has initial vertex in  $H_0$  and terminal vertex in  $H_1$  and that  $E_2$  is a closed path with basepoint in  $H_0$ . For  $j = 0, 1$  let  $\tau_j$  be a loop based at the unique vertex of  $H_j$ . Let  $\sigma_0 = \tau_0$ ,  $\sigma_1 = E_2$  and  $\sigma_2 = E_1 \tau_1 \overline{E_1}$ .

The remaining case is that  $H$  has two components  $H_0$  and  $H_1$  and neither  $E_1$  nor  $E_2$  is a loop. We may assume that both  $E_1$  and  $E_2$  have initial vertex in  $H_0$  and terminal vertex in  $H_1$ . The argument in this case is a variation of the one used in the three preceding cases. For  $j = 0, 1$  let  $\tau_j$  be a loop based at the unique vertex of  $H_j$ . Let  $\sigma_0 = \tau_0$ , let  $\sigma_1 = E_1 \tau_1 \overline{E_1}$  and let  $\sigma_2 = E_1 \overline{E_2}$ . Define  $a, b$  and  $c$  from  $\sigma_0, \sigma_1$  and  $\sigma_2$  as above. Then  $[H] = \{[a], [b]\}$ ,  $[H \cup E_1] = [\langle a, b \rangle]$  and  $[H \cup E_2] = [\langle a, \bar{c}bc \rangle]$ .

Choose  $m$  so that  $[ab^m], [cb^m]$  and  $[a^m \bar{c}bc]$  are not in  $\mathcal{A}_{\text{na}}\Lambda^\pm$ . Consider the rank two free factors

$$\langle a, b \rangle \quad \langle ab^m, \bar{c}b^m \rangle \quad \langle \bar{c}b^m, a^m \bar{c}bc \rangle \quad \langle a, \bar{c}bc \rangle$$

whose consecutive intersections contain

$$ab^m \quad \bar{c}b^m \quad a^m \bar{c}bc$$

respectively. Three applications of Lemma 4.14 (3) show that  $|W(S_2) - W(S_1)| \leq 6M$ .  $\square$

The following corollary puts the pieces together to finish the proof of Theorem 1.1 (1):

**Corollary 4.21.** *Suppose that  $\phi \in \text{Out}(F_n)$ . If some  $\Lambda^+ \in \mathcal{L}(\phi)$  fills then the action of  $\phi$  on  $\mathcal{FS}(F_n)$  is loxodromic.*

*Proof.* After replacing  $\phi$  by an iterate we may assume that  $\phi$  and  $\phi^{-1}$  are rotationless. For any one-edge splitting  $S$ , we have  $W(S^{\phi^m}) = W(S) - m$  by Remark 4.19. Lemma 4.20 therefore implies that the distance between  $S^{\phi^m}$  and  $S$  in  $\mathcal{FS}(F_n)$  grows linearly in  $m$ , which completes the proof of the corollary.  $\square$

**Remark 4.22.** As said in the introduction, it is known from [BF11] Theorem 9.3 that  $\phi \in \text{Out}(F_n)$  acts loxodromically on the free factor complex if and only if  $\phi$  is fully irreducible. The “if” direction follows from the same method of proof as in Corollary 4.21, but with a shorter argument using Lemma 4.16 in place of Lemma 4.20 and using that every nonfilling conjugacy class is not carried by  $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$ .

#### 4.5 Axes with distinct ends: a part of Theorem 1.2.

The following statement, ‘part’ of the proof of Theorem 1.2, is proved here since it is a corollary to the methods of Section 4.4.

**Corollary 4.23.** *Given  $\phi, \psi \in \text{Out}(F_n)$  and filling lamination pairs  $\Lambda_\phi^\pm \in \mathcal{L}^\pm(\phi)$  and  $\Lambda_\psi^\pm \in \mathcal{L}^\pm(\psi)$ , if  $\Lambda_\phi^+ \neq \Lambda_\psi^+$  then  $\partial_-\phi \neq \partial_-\psi$ .*

*Proof.* It suffices to show that for any one-edge free splitting  $T = \langle G, H \rangle$  with corresponding marked graph pair  $(G, H)$ , and for all positive constants  $D$ , there exists a positive constant  $L$  so that the distance in  $\mathcal{FS}(F_n)$  between  $T^{\phi^{-k}}$  and  $T^{\psi^{-l}}$  is  $\geq D$  for all  $k, l > L$ .

Assume the notation of Lemma 4.14 applied to  $\phi$ . Thus  $U_\phi^\pm$  are attracting neighborhoods of generic leaves  $\gamma_\phi^\pm$  of  $\Lambda_\phi^\pm$ ,  $M_\phi$  is a positive constant and  $w_\phi$  is a function defined on conjugacy classes not carried by the non-attracting subgroup system  $\mathcal{A}_\phi$  associated to  $\Lambda_\phi^\pm$ . Let  $W_\phi$  be the corresponding function defined on one edge splittings in Definition 4.15.

We will prove that there is an upper bound for  $W_\phi(\psi^l(\mathcal{F}(T)))$  that is independent of  $l \geq 0$ . To see why this suffices, note that  $\mathcal{F}(T^{\phi^{-k}}) = \phi^k(\mathcal{F}(T))$  and that  $\mathcal{F}(T^{\psi^{-l}}) = \psi^l(\mathcal{F}(T))$  by Remark 4.19. Since  $W_\phi(\phi^k(\mathcal{F}(T))) = W_\phi(\mathcal{F}(T)) + k$ , we have that

$$\left| W_\phi(T^{\phi^{-k}}) - W_\phi(T^{\psi^{-l}}) \right| = \left| W_\phi(\phi^k(\mathcal{F}(T))) - W_\phi(\psi^l(\mathcal{F}(T))) \right| \rightarrow \infty \quad \text{as } l, k \rightarrow \infty$$

and so Lemma 4.20 completes the proof.

Lemma 4.13 implies that  $B_1 = \{\gamma_\phi^+\}$  and  $B_2 = \{\gamma_\psi^+\}$  satisfy the hypotheses of Proposition 3.1 of [HM13c]. The conclusion of that proposition is the existence of neighborhoods  $V_\phi^+$  of  $\gamma_\phi^+$  and  $V_\psi^+$  of  $\gamma_\psi^+$  so that no proper free factor system carries both a conjugacy class carried by  $V_\phi^+$  and a conjugacy class carried by  $V_\psi^+$ . Choose  $N_\phi > 0$  so that  $\phi^{N_\phi}(U_\phi^+) \subset V_\phi^+$ .

By Lemma 4.11 (1),  $\mathcal{F}(T)$  carries a conjugacy class  $[a]$  that is not carried by the non-attracting subgroup system  $\mathcal{A}_\psi$  associated to  $\Lambda_\psi^\pm$ . Choose  $N_\psi > 0$  so that  $\psi^l([a])$  is carried by  $V_\psi^+$  for all  $l \geq N_\psi$ ; since  $\psi^l(\mathcal{F}(T))$  carries  $\psi^l([a])$ , it can not carry any conjugacy class that is carried by  $V_\phi^+$ . It follows that  $\phi^{-N_\phi}\psi^l(\mathcal{F}(T))$  does not carry any conjugacy class that is carried by  $U_\phi$ ; but Lemma 4.11 (1) implies that  $\phi^{-N_\phi}\psi^l(\mathcal{F}(T))$  carries a conjugacy class that is not carried by  $\mathcal{A}_\phi$ , and so Remark 4.17 implies that

$$W_\phi(\phi^{-N_\phi}\psi^l(\mathcal{F}(T))) \leq M_\phi$$

It follows that

$$W_\phi(\psi^l(\mathcal{F}(T))) \leq M_\phi + N_\phi$$

for all  $l \geq N_\psi$  by Remark 4.18. □

## 5 Proof of Theorem 1.4: The expansion factor kernel $K$

We assume throughout this section that  $\eta \in \text{IA}_n(\mathbb{Z}/3)$  is rotationless and that  $\Lambda_\eta^+ \in \mathcal{L}(\eta)$  is filling. Let  $\text{Ker}(\text{PF})$  denote the kernel of  $\text{PF} = \text{PF}_{\Lambda_\eta^+} : \text{Stab}(\Lambda_\eta^+) \rightarrow \mathbb{R}$ , and let

$$K = \text{Ker}(\text{PF}) \cap \text{IA}_n(\mathbb{Z}/3) = \text{Ker}(\text{PF} \mid \text{Stab}(\Lambda_\eta^+) \cap \text{IA}_n(\mathbb{Z}/3))$$

Let  $\mathcal{F}_{\text{ng}}$  be the smallest free factor system that carries every non-generic line of  $\Lambda_\eta^+$ .

### 5.1 Attracting laminations of $K$

The next result shows that every attracting lamination of every element of  $K$  is a geometric lamination, and gives explicit description of the behavior of generic leaves of  $\Lambda_\eta^+$  in CTs representing any element of  $K$ .

**Lemma 5.1.** *If  $\phi \in K$  is rotationless, and if  $f : G \rightarrow G$  is a CT representing  $\phi$  in which  $\mathcal{F}_{\text{ng}}$  is realized by a proper core filtration element  $G_r$ , then the highest stratum of  $f : G \rightarrow G$  is NEG, and each EG stratum is geometric (equivalently each element of  $\mathcal{L}(\phi)$  is geometric). Furthermore, if  $\gamma_\eta$  is the realization in  $G$  of a generic leaf of  $\Lambda_\eta^+$  then  $\gamma_\eta$  splits into a concatenation of Nielsen paths for  $f$  and is carried by the non-attracting subgroup system of each element of  $\mathcal{L}(\phi)$ .*

**Remark 5.2.** Since  $\gamma$  is birecurrent, it splits into a concatenation of Nielsen paths for  $f$  if and only if it is carried by  $[\text{Fix}(\Phi)]$  for some automorphism  $\Phi$  representing  $\phi$ . This allows one to drop the hypothesis in Lemma 5.1 that  $\mathcal{F}_{\text{ng}}$  is realized by a core filtration element. We make no use of this so leave the details to the interested reader.

*Proof.* We show first that the highest stratum  $H_N$  of  $G$  is NEG by assuming that  $H_N$  is EG with associated lamination pair  $\Lambda_\phi^\pm$  and arguing to a contradiction. Proposition 3.3.3 of [BFH00] and the assumption that  $\text{PF}_{\Lambda_\eta^+}(\phi) = 0$  imply that  $\Lambda_\eta^+$  and  $\Lambda_\eta^-$  are not elements of  $\mathcal{L}(\phi)$  and in particular are not equal to  $\Lambda_\phi^+$ . By construction,  $\Lambda_\eta^+$  and  $\Lambda_\eta^-$  are not carried by  $\mathcal{F}_{\text{ng}}$  and so are not proper sublaminations of  $\Lambda_\eta^+$ . Since  $\Lambda_\eta^+$  is preserved by  $\phi$ , it contains every line to which  $\gamma_\eta$  is weakly attracted. Thus  $\gamma_\eta$  is not weakly attracted to  $\Lambda_\phi^\pm$ . Corollary 2.1 of [HM13c] implies that  $\gamma_\eta$  is carried by  $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$ . But this is impossible: if  $\Lambda_\phi^\pm$  is non-geometric then  $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$  is a proper free factor system, whereas if  $\Lambda_\phi^\pm$  is geometric then  $\mathcal{A}_{\text{na}}(\Lambda_\phi^\pm)$  consists of a proper free factor system plus a single rank 1 component, but  $\gamma_\eta$  is not periodic and is not carried by any proper free factor system. We have now reached the desired contradiction and so have proven that  $H_N$  is NEG.

**Remark 5.3.** In the case that  $\phi$  is UPG, it is obvious that  $H_N$  is NEG so the above paragraph is unnecessary. In what remains of the proof, the only properties of  $f$  that we use are that it satisfy the basic splitting property for NEG edges and that every periodic Nielsen path has period one. Hence the lemma holds for UPG  $\phi$  with these weaker assumptions on  $f : G \rightarrow G$ . This will be used in the proof of Lemma 5.8.

Let  $E_N$  be the unique oriented edge in  $H_N$ . By the basic splitting property for NEG edges, subdividing  $\gamma_\eta$  at the initial vertex of each copy of  $E_N$  that it crosses and at the



terminal vertex of each copy of  $\overline{E}_N$  that it crosses defines a splitting of  $\gamma_\eta$ . The birecurrence of  $\gamma_\eta$  implies that this splitting is bi-infinite  $\gamma_\eta = \dots \cdot \gamma_{-1} \cdot \gamma_0 \cdot \gamma_1 \cdot \dots$ . For each  $i$ , the endpoints of  $\gamma_i$  are fixed and so either  $\gamma_i$  is a periodic, and hence fixed, Nielsen path or the combinatorial length  $|f_\#^k(\gamma_i)| \rightarrow \infty$  as  $k \rightarrow \infty$ . We assume that some, and hence infinitely many,  $\gamma_i$ 's are not Nielsen paths and argue to a contradiction.

Choose  $s < t$  so that neither  $\gamma_s$  nor  $\gamma_t$  is a Nielsen path and such that  $\gamma_i$  crosses  $E_N$  (in either direction) for some  $s < i < t$ . Then  $f_\#^k(\gamma_s \cdot \dots \cdot \gamma_t) = \alpha_k E_N \beta_k$  or  $\alpha_k \overline{E}_N \beta_k$  where:

- (1)  $|\alpha_k|, |\beta_k| \rightarrow \infty$ .
- (2)  $\alpha_k$  and  $\beta_k$  cross  $E_N$  (in either direction) a uniformly bounded number of times.

By taking limits based at the central  $E_N$  or  $\overline{E}_N$  we find a line in  $\Lambda_\eta^+$  that crosses  $E_N$  (in either direction) a finite non-zero number of times. Such a line would be non-generic but not contained in  $G_r$  which is the desired contradiction. This completes the proof that  $\gamma_\eta$  is a concatenation of Nielsen paths. In particular,  $\Lambda_\eta^+$  does not carry any element of  $\mathcal{L}(\phi)$  and hence is not weakly attracted to any element of  $\mathcal{L}(\phi)$ . As above, Corollary 2.17 of [HM13c] implies that  $\Lambda_\eta^+$  is carried by the non-attracting subgroup system of each element of  $\mathcal{L}(\phi)$ . Since  $\Lambda_\eta^+$  fills, Theorem F of [HM13c] implies that each element of  $\mathcal{L}(\phi)$  is geometric.  $\square$

Because of Lemma 5.1, henceforth in this section we will use without comment the fact that each attracting lamination of each element of  $K$  is geometric.

**Corollary 5.4.** *For each  $\phi \in K$  and  $\Lambda \in \mathcal{L}(\phi)$  the nonattracting subgroup system  $\mathcal{A}_{na}\Lambda$  fills  $F_n$ .*

*Proof.* We may pass to a rotationless power of  $\phi$  and choose a representative CT  $f: G \rightarrow G$  in which  $\mathcal{F}_{ng}$  is realized by a filtration element. By Lemma 5.1, the filling line  $\gamma_\eta$  is carried by  $\mathcal{A}_{na}\Lambda$ .  $\square$

## 5.2 UPG subgroups of $K$

The following proposition, which is a piece of Theorem 1.4, is the main result of this section. Its proof appears at the end of the section.

**Proposition 5.5.** *Every UPG subgroup  $\mathcal{H} < K$  is abelian, linear and finitely generated.*

Recall from Definition 4.1.3 of [BFH00] that if  $f: G \rightarrow G$  is a topological representative of  $\phi$ , if the NEG stratum  $H_i$  is a single edge  $E_i$  and if  $\alpha$  is a path in  $G_{i-1}$  then the paths  $E_i \gamma, \gamma \overline{E}_i$  and  $E_i \gamma \overline{E}_i$  are *basic paths of height  $i$* . By the basic splitting property for NEG edges, each path of height  $i$  with endpoints at vertices has a canonical splitting, called the *highest edge splitting*, into a concatenation of basic subpaths of height  $i$  and subpaths in  $G_{i-1}$ .

Consider a marked graph  $G$  equipped with a filtration  $\emptyset = G_0 \subset G_1 \subset \dots \subset G_N = G$  by subgraphs in which each stratum  $G_i - G_{i-1}$  is a single oriented edge  $E_i$ . Following [BFH05] we say that a homotopy equivalence  $f: G \rightarrow G$  is *upper triangular* if for each  $i$  we have  $f(E_i) = \overline{v}_i E_i u_i$  for some (possibly trivial) closed paths  $u_i = u_i(f), v_i = v_i(f) \subset G_{i-1}$ . Let  $\mathcal{Q}(G)$  be the set of upper triangular homotopy equivalences of  $G$  up to homotopy relative to the vertices of  $G$ , which is a group under the operation of composition (Lemma 6.1 of [BFH05]). There is a natural homomorphism  $\mathcal{Q}(G) \mapsto \text{Out}(F_n)$ .

**Example 5.6.** Here is a special case of an outer automorphism  $\phi$  as considered in Example 4.1. Continuing the notation of that example, let  $G_1 \subset G$  be roses of rank  $m = 3$  and rank 5, respectively. Let  $X, Y, Z$  be the edges of  $G_1$  and  $A, B$  the edges of  $H_2 = G \setminus G_1$ . Fix a nontrivial word  $w$  in  $\langle X, YX\bar{Y} \rangle$ , and let  $\mathcal{H}'$  be the subgroup of  $\mathcal{Q}(G)$  whose elements have the form

$$X \mapsto X \quad Y \mapsto YX^{3i} \quad Z \mapsto Zw^{3j} \quad A \mapsto A \quad B \mapsto B$$

Then  $\mathcal{H}'$  is a rank two abelian linear subgroup and every word in  $\langle X, YX\bar{Y}, Zw\bar{Z} \rangle$  is fixed by every element of  $\mathcal{H}'$ . The smallest free factor that contains  $\langle X, YX\bar{Y} \rangle$  equals  $\langle X, Y \rangle$  because it is contained in  $\langle X, Y \rangle$  and properly contains  $\langle X \rangle$ . Similarly the smallest free factor that contains  $\langle X, YX\bar{Y}, Zw\bar{Z} \rangle$  equals  $\langle X, Y, Z \rangle$ . Choose a word  $\sigma \in \langle X, YX\bar{Y}, Zw\bar{Z} \rangle$  that fills  $\langle X, YX\bar{Y}, Zw\bar{Z} \rangle$  and hence fills  $\langle X, Y, Z \rangle$ . Using this  $\sigma$ , let  $\phi \in \text{Out}(F_5)$  and  $\Lambda \in \mathcal{L}(\phi)$  be as in Example 4.1. Then  $\mathcal{H}'$  injects into  $\text{Out}(F_5)$  producing a rank two linear subgroup  $\mathcal{H}$  of  $K$ .

The following lemma gathers some earlier results regarding finitely generated UPG subgroups, and Lemma 5.8 isolates the key additional property satisfied by such subgroups of  $\mathcal{K}$ .

**Lemma 5.7.** *Suppose that  $\mathcal{H}$  is a finitely generated UPG subgroup and that  $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_k = \{[F_n]\}$  is a maximal  $\mathcal{H}$ -invariant filtration by properly nested free factor systems, one of which is  $\mathcal{F}_{ng}$ . Then there exists a marked graph  $G$ , a filtration  $\emptyset = G_0 \subset G_1 \subset \dots \subset G_N = G$  and a lift of  $\mathcal{H}$  to a subgroup  $\mathcal{H}'$  of  $\mathcal{Q}(G)$  with the following properties.*

- (1) *Each  $\mathcal{F}_i$  is realized by a core filtration element.*
- (2) *If  $i < j$  and  $G_i$  and  $G_j$  are consecutive core filtration elements then either:  $j = i + 1$  and  $G_j \setminus G_i$  is a single edge  $E_{i+1}$  forming a loop that is disjoint from  $G_i$ ; or  $j = i + 2$  and  $G_j \setminus G_i$  is a single topological arc subdivided into two oriented edges  $E_{i+1}$  and  $E_{i+2}$  with a common initial endpoint not in  $G_i$  and with terminal endpoints in  $G_i$ .*

Furthermore, for each  $f : G \rightarrow G$  in  $\mathcal{H}'$  the following hold:

- (3) *If  $C$  is a component of the union of the fixed edges of  $f$  and if  $C$  is a topological circle then at least one vertex in  $C$  is the basepoint of at least three fixed directions.*
- (4) *Every periodic Nielsen path for  $f$  with endpoints at vertices has period one.*

*Proof.* A construction of  $\mathcal{H}'$  satisfying items (1) - (3) is given on the last page of [BFH05]. Since there are no EG strata, item (3) implies that each vertex is a principal fixed point as defined in Definition 3.18 of [FH11]. If an element  $f : G \rightarrow G$  of  $\mathcal{H}'$  satisfies the conclusions of Theorem 2.19 of [FH11] then it also satisfies item (4) by Lemma 3.28 of [FH11]. The only conclusion of Theorem 2.19 that might fail is item (P) which prohibits the existence of ‘extraneous’ fixed edges. This item (P) fails precisely if for some  $G_i$  one of the non-loop edges  $E_{i+1}, E_{i+2}$  occurring in (2) is fixed. If this happens we can still show that  $f : G \rightarrow G$  satisfies (4) by modifying it, collapsing one of  $E_{i+1}, E_{i+2}$  which is fixed. Doing this for each  $i$  as necessary, the new homotopy equivalence  $f' : G' \rightarrow G'$  (see page 7 of [BH92]) represents  $\phi$  and satisfies the conclusions of Theorem 2.19 and so each of its periodic Nielsen paths has

period one. But the image in  $G'$  of a periodic Nielsen path for  $f$  with period greater than one is a periodic Nielsen path for  $f'$  with period greater than one. It follows that there are no such periodic Nielsen paths for  $f$  and hence that  $f$  satisfies (4).  $\square$

If  $\mathcal{H}'$  is a subgroup of  $\mathcal{Q}(G)$  then we say that an edge or a point in  $G$  is *universally fixed* if it is fixed by each  $f \in \mathcal{H}'$  and that a path in  $G$  with endpoints at vertices is *universally Nielsen* if it is a Nielsen path for each  $f \in \mathcal{H}'$ . A universal Nielsen path is an *indivisible universal Nielsen path* if it cannot be written as a concatenation of two non-trivial universal Nielsen paths. Note that an indivisible universal Nielsen path need not be an indivisible Nielsen path for each  $f \in \mathcal{H}$ .

**Lemma 5.8.** *Suppose that  $\mathcal{H}$  is a finitely generated UPG subgroup of  $K$ . Then there is a maximal  $\mathcal{H}$ -invariant filtration  $\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_k = \{[F_n]\}$  by properly nested free factor systems, a marked graph  $G$ , a filtration  $\emptyset = G_0 \subset G_1 \subset \dots \subset G_N = G$  and a lift of  $\mathcal{H}$  to a subgroup  $\mathcal{H}'$  of  $\mathcal{Q}(G)$  satisfying the conclusions of Lemma 5.7 plus the following additional property.*

(5) *For each edge  $E_i$  either:*

- (a)  *$E_i$  is universally fixed; or*
- (b) *there is a universal closed Nielsen path  $w_i$  such that  $f(E_i) = E_i w_i^{d_i(f)}$  for all  $f \in \mathcal{H}'$  and some  $d_i(f) \in \mathbb{Z}$ . Moreover,  $\{E_i w_i^p \bar{E}_i : p \neq 0\}$  are the only indivisible universal Nielsen paths of height  $i$ .*

*Proof.* Choose a maximal  $\mathcal{H}$ -invariant filtration  $\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_k = \{[F_n]\}$  by properly nested free factor systems and then apply Lemma 5.7 to produce  $\mathcal{H}' < \mathcal{Q}(G)$ . We will modify  $G$  and  $\mathcal{H}'$  by downward induction to arrange that (5) is satisfied.

Let  $\mathcal{C}$  be the set of universal Nielsen paths. We make frequent use of the fact that highest edge subdivision decomposes an element of  $\mathcal{C}$  with height  $j$  into a concatenation of elements of  $\mathcal{C}$ , each of which is either a basic path of height  $j$  or a path with height strictly less than  $j$ . In particular, an indivisible element of  $\mathcal{C}$  of height  $j$  is a basic path of height  $j$ .

The maximality assumption on  $\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_k = \{[F_n]\}$  is used to establish a piece of (b); namely, that if  $E_j$  is not universally fixed then each closed indivisible  $\alpha \in \mathcal{C}$  with height  $j$  has the form  $E_j \mu \bar{E}_j$  for some  $\mu$  with height less than  $j$ . As noted above,  $\alpha$  is a basic path of height  $j$ . We therefore need only prove that if  $\alpha$  is closed and if  $\alpha = E_j \mu$  for some  $\mu$  with height  $< j$  then  $E_j$  is universally fixed and  $\mu$  is trivial. There is a basis element  $a \in F_n$  such that  $[a]$  is the conjugacy class determined by  $\alpha$ . Moreover, if  $m$  is the smallest index for which  $\mathcal{F}_m$  carries  $[a]$ , then  $\mathcal{F}_{m-1} \cup \{[a]\}$  is an  $\mathcal{H}$ -invariant free factor system that properly contains  $\mathcal{F}_{m-1}$  and is contained in  $\mathcal{F}_m$ . It follows that  $\mathcal{F}_m = \mathcal{F}_{m-1} \cup [a]$  and hence that  $[a]$  is represented by a loop component of  $G_i$ . It then follows from Lemma 5.7-(2) that  $E_i$  is that loop, and it is disjoint from  $G_{i-1}$ , so it is universally fixed and  $\mu$  is trivial.

The assumption that  $H < K$  is used to prove that every edge of  $G$  is crossed by some element of  $\mathcal{C}$ . Highest edge subdivision decomposes  $\gamma_\eta$ , which crosses every edge of  $G$ , into a concatenation of paths  $\gamma_l$  based at the initial vertex of the highest edge in  $G$ . Each  $f : G \rightarrow G$  in  $\mathcal{H}'$  satisfies the basic splitting property of NEG edges and each periodic indivisible Nielsen path has period one. Lemma 5.1 and Remark 5.3 therefore imply that each  $\gamma_l$  is a universal Nielsen path and hence an element of  $\mathcal{C}$ .

We now turn to the induction argument, assuming that (5) is satisfied by all  $j > i$  (which is obvious if  $i$  is the maximal height) and proving that  $E_i$  satisfies (5).

**Claim 1 : Some element of  $\mathcal{C}$  has height  $i$ .**

To verify claim 1 we assume that  $E_i$  is crossed by  $\alpha \in \mathcal{C}$  with height  $k > i$  and prove that  $E_i$  is crossed by some  $\alpha' \in \mathcal{C}$  with height  $< k$ . There is no loss in assuming that  $\alpha$  is an indivisible element of  $\mathcal{C}$  and that  $E_i \not\subset \mathcal{C}$ . The moreover part of (b) therefore implies that  $\alpha = E_k w_k^p \tilde{E}_k$  for some  $p \neq 0$  so  $E_i$  is crossed by  $\alpha' = w_k \in \mathcal{C}$  and the first claim is verified.

Choose a lift  $\tilde{x}_i$  of the initial endpoint  $x_i$  of  $E_i$  in the universal cover  $\tilde{G}$  of  $G$ , let  $\tilde{E}_i$  be the lift of  $E_i$  with initial vertex  $\tilde{x}_i$  and let  $\Gamma_{i-1} \subset \tilde{G}$  be the component of the full pre-image of  $G_{i-1}$  that contains the terminal endpoint  $\tilde{y}_i$  of  $\tilde{E}_i$ . For each  $f \in \mathcal{H}'$ , let  $\tilde{f} : \tilde{G} \rightarrow \tilde{G}$  be the lift of  $f$  that fixes  $\tilde{x}_i$  and note that  $\Gamma_{i-1}$  is  $\tilde{f}$ -invariant. Let  $\mathcal{T}_i$  be the set of covering translations of  $\tilde{G}$  that commute with  $\tilde{f}$  for each  $f \in \mathcal{H}'$  and whose axis is contained in  $\Gamma_{i-1}$ .

**Claim 2: Either there is a point in  $\Gamma_{i-1}$  that is universally fixed or  $\mathcal{T}_i \neq \emptyset$ .**

By the claim 1, there exists a basic path  $\alpha \in \mathcal{C}$  with height  $i$ . Lift  $\alpha$  to a path  $\tilde{\alpha}$  with initial endpoint  $\tilde{x}_i$  and terminal endpoint  $\tilde{z}$  and note that  $\tilde{z}$  is fixed by each  $\tilde{f}$ . If  $\alpha$  is not a closed path then  $\tilde{z} \in \Gamma_{i-1}$  and we are done. Assuming then that  $\alpha$  is closed, let  $T : \tilde{G} \rightarrow \tilde{G}$  be the covering translation that carries  $\tilde{x}_i$  to  $\tilde{z}$ . Then  $T\tilde{f}(\tilde{x}_i) = T(\tilde{x}_i) = \tilde{z} = \tilde{f}(\tilde{z}) = \tilde{f}T(\tilde{x}_i)$ . Since  $T\tilde{f}$  and  $\tilde{f}T$  agree on a point they are equal and so  $T$  commutes with  $\tilde{f}$ . The axis  $A_\alpha$  of  $T$  is contained in  $\Gamma_{i-1}$  because  $\alpha = E_i \mu \tilde{E}_i$  for some  $\mu \subset G_{i-1}$ . Thus  $T \in \mathcal{T}_i$  and the second claim is verified.

**Claim 3: If there are no universal fixed points in  $\Gamma_{i-1}$  then  $A(T) = A(T')$  for all  $T, T' \in \mathcal{T}_i$ .**

The axis  $A(T)$  of  $T \in \mathcal{T}_i$  is  $\tilde{f}_\#$ -invariant for all  $f \in \mathcal{H}'$ . The  $f_\#$ -image of the highest edge splitting of  $A(T)$  is the highest edge splitting of  $f_\#(A(T)) = A(T)$  so  $f_\#$  preserves the set  $\mathcal{V}(T)$  of highest edge splitting vertices of  $A(T)$ . Identify  $\mathcal{V}(T)$  with  $\mathbb{Z}$  by considering the order induced from  $A(T)$ . Then each  $\tilde{f}$  acts on  $\mathcal{V}(T)$  by a (possibly trivial) translation.

Choose a root free  $T_0 \in \mathcal{T}_i$  so that the axis  $A(T_0)$  has maximal height. To verify the claim, we suppose that some  $\tilde{f}'$  acts as a non-trivial translation on  $\mathcal{V}(T_0)$  and that there exists  $T_1 \in \mathcal{T}_i$  with  $A(T_1) \neq A(T_0)$  and argue to a contradiction. Under iteration by  $\tilde{f}'$ , the elements of  $\mathcal{V}(T_0)$  converge to some  $Q \in \partial\Gamma_{i-1}$ . The elements of  $\mathcal{V}(T_1)$  have two accumulation points in  $\partial\Gamma_{i-1}$ ; choose one,  $P$ , so that the elements of  $\mathcal{V}(T_1)$  are not moving away from  $P$  under iteration by  $\tilde{f}'$ . The line  $\tilde{L}$  with endpoints  $P$  and  $Q$  is  $\tilde{f}'_\#$ -invariant so its highest edge splitting vertices  $\mathcal{V}(\tilde{L})$  are preserved by  $\tilde{f}'$ . Since the height of  $A(T_0)$  is greater than or equal to the height of  $A(T_1)$ , we have  $\mathcal{V}(T_0) \cap \tilde{L} \subset \mathcal{V}(\tilde{L})$  and  $\tilde{f}'$  moves the elements of  $\mathcal{V}(\tilde{L})$  toward  $Q$ . But this implies that both ends of  $\tilde{L}$  have the same height and that the elements of  $\mathcal{V}(T_1) \cap \tilde{L} \subset \mathcal{V}(\tilde{L})$  are moved away from  $P$ . This contradiction completes the proof of the third claim.

We are now able to complete the proof in two special cases. In the first,  $\tilde{y}_i$  is universally fixed, which means that  $E_i$  is universally fixed and (a) is satisfied. In the second, there are no universal fixed points in  $\Gamma_{i-1}$  and  $\tilde{y}_i \in \mathcal{V}(T)$  for some (and hence every) root-free  $T_0$  in  $\mathcal{T}_i$ . Let  $\tilde{w}_i$  be the path from  $\tilde{y}_i$  to  $T_0(\tilde{y}_i)$  and let  $w_i$  be the projected image of  $\tilde{w}_i$ . If

$\tilde{f}$  acts by a non-trivial translation on  $\mathcal{V}(T_0)$  then the paths  $\tilde{\rho}_j$  connecting  $\tilde{f}^j(\tilde{y}_i) \in \mathcal{V}(T_0)$  to  $\tilde{f}^{j+1}(\tilde{y}_i) \in \mathcal{V}(T_0)$  have uniformly bounded length. Since  $\tilde{f}_\#$  permutes these paths, the projection  $\rho_j$  of  $\tilde{\rho}_j$  is a periodic Nielsen path and hence a Nielsen path  $\rho$  that is independent of  $j$ . Since the  $\tilde{\rho}_j$ 's tile a terminal ray of  $A(T)$ , there exists  $d_i(f) \in \mathbb{Z}$  such that  $\rho = w_i^{d_i(f)}$ . By construction,  $f(E_i) = E_i\rho$  so we have proved that  $f(E_i) = E_iw_i^{d_i(f)}$ .

For the moreover part of (b), suppose that  $\alpha \in \mathcal{C}$  has height  $i$  and is indivisible. Let  $\tilde{\alpha}$  be the lift of  $\alpha$  with initial endpoint  $\tilde{x}$ . As noted above,  $\alpha$  is a basic path of height  $i$ . If  $\alpha = E_i\mu$  for some  $\mu \subset G_{i-1}$  then the terminal endpoint  $\tilde{z}$  of  $\tilde{\alpha}$  is a universal fixed point in  $\Gamma_{i-1}$  which contradicts the assumptions of this special case. Thus  $\alpha = E_i\mu\bar{E}_i$  for some  $\mu \subset G_{i-1}$ . The covering translation  $T$  that maps  $\tilde{x}_i$  to  $\tilde{z}$  is an element of  $\mathcal{T}_i$  and so  $T = T_0^p$  for some  $p \neq 0$ . Since  $\mu$  lifts to the path connecting  $\tilde{y}_i \in \mathcal{V}(T_0)$  to  $T(\tilde{y}_i) = T_0^p(\tilde{y}_i) \in \mathcal{V}(T_0)$ , it follows that  $\mu = w_i^p$  as desired.

For the general case, we modify  $G$  by sliding the terminal end of  $E_i$  as described on pages 579 - 581 of in section 5.4 of [BFH00]. Choose a point  $\tilde{v} \in \Gamma$  as follows. If possible, choose  $\tilde{v}$  to be universally fixed. Otherwise, choose a root-free  $T_0 \in \mathcal{T}_i$  and choose  $\tilde{v} \in \mathcal{V}(T_0)$ . Now choose a path  $\tilde{\sigma}$  (which is trivial in the special case) from  $\tilde{y}_i$  to  $\tilde{v}$  and let  $\sigma$  be its projected image in  $G_{i-1}$ . One slides  $E_i$  along  $\sigma$  by identifying a proper initial segment of  $\bar{E}_i$  with  $\sigma$ . The effect is that  $E_i$  is replaced by an edge  $E'_i$  whose terminal endpoint is  $\tilde{v}$ . This folding operation produces a new marked graph  $G'$  and homotopy equivalences  $p : G \rightarrow G'$  and  $p' : G' \rightarrow G$  that are the 'identity' on the common edges of  $G$  and  $G'$ , that are homotopy inverses of each other and that preserve the markings.

For each  $f \in \mathcal{Q}(G)$  there is an  $f' \in \mathcal{Q}(G')$  such that  $f'(E') = (pf p')_\#(E')$  for each edge  $E'$  of  $G'$ . This induces an isomorphism from  $\mathcal{H}' < \mathcal{Q}(G)$  to a subgroup of  $\mathcal{Q}(G')$ , which we continue to call  $\mathcal{H}'$ . The new  $\mathcal{H}'$  continues to satisfy items (1) - (3) of Lemma 5.7. It is shown in the last paragraph on page 580 of [BFH00] that  $p_\#$  induces a period preserving bijection between the periodic Nielsen paths of  $f$  and the periodic Nielsen paths of  $f'$ . This implies (4) and implies that the inductive hypothesis still applies to our new  $\mathcal{Q}'$ . As discussed on the top of page 581, the lift of  $f'$  to  $\tilde{f}'$  is still defined and  $\tilde{f}' \mid \Gamma_{i-1} = \tilde{f} \mid \Gamma_{i-1}$ . The positive effect for us is that the terminal end of  $E'_i$  is now  $\tilde{v}$  and we are reduced to one of the two special case. The proofs of all these assertions are routine applications of sliding and are left to the reader. This completes the proof of Lemma 5.8.  $\square$

*Proof of Proposition 5.5.* A subgroup is abelian if all of its finitely generated subgroups are. By [CV86], each abelian subgroup of  $\text{Out}(F_n)$  is finitely generated. It therefore suffices to show that each finitely generated subgroup  $\mathcal{H}_0$  of  $\mathcal{H}$  is abelian and its elements are all linearly growing. Applying Lemma 5.8 (5) to  $\mathcal{H}_0$  implies that the elements of  $\mathcal{H}'_0$  commute with each other. Since  $\mathcal{H}_0$  is isomorphic to  $\mathcal{H}'_0$ , it is also abelian. Lemma 5.8 (5) also implies that each element of  $\mathcal{H}_0$  is linear.  $\square$

### 5.3 Ordering attracting laminations

Let  $\mathcal{L}(F_n)$  be the set of ordered pairs  $(\Lambda, \phi)$  such that  $\phi \in \text{Out}(F_n)$  and  $\Lambda \in \mathcal{L}(\phi)$ . We often abbreviate the ordered pair using subscript notation  $\Lambda_\phi$ ; the meaning should be clear by context. When  $\phi$  is understood we abbreviate further and write simply  $\Lambda$ .

Let  $\mathcal{F} = \{[F]\}$  be a  $\phi$ -invariant free factor system that supports  $\Lambda$  and that has a

single component  $[F]$  (e.g. the free factor support of  $\Lambda$ , but we shall need to consider more general  $\mathcal{F}$ ). There is an associated outer automorphism  $\phi \mid \mathcal{F} \in \text{Out}(F)$  and there is an associated injection from  $\mathcal{L}(\phi \mid \mathcal{F})$  to  $\mathcal{L}(\phi)$ . Let  $\Lambda_\phi \mid \mathcal{F}$  denote the element of  $\mathcal{L}(\phi \mid \mathcal{F})$  that corresponds to  $\Lambda_\phi$ . In the free factor  $F$  we have the vertex group system  $\mathcal{A}_{\text{na}}(\Lambda_\phi \mid \mathcal{F})$ , which we may also regard as a vertex group system in  $F_n$ .

The proof of Theorem 1.4 is structured in part as an induction based on the following partial ordering on elements of  $\mathcal{L}(F_n)$ .

**Notation 5.9.** We write  $\Lambda_\phi < \Lambda_\psi$  if either

- (1)  $\mathcal{F}(\Lambda_\phi)$  is properly contained in  $\mathcal{F}(\Lambda_\psi)$  or
- (2)  $\mathcal{F}(\Lambda_\phi) = \mathcal{F}(\Lambda_\psi)$  and, letting  $\mathcal{F}$  denote this subgroup system,  $\mathcal{A}_{\text{na}}(\Lambda_\phi \mid \mathcal{F})$  is properly contained in  $\mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F})$

**Remark 5.10.** Regarding item (2), in the special case  $\mathcal{F} = \{[F_n]\}$  one knows that  $\mathcal{A}_{\text{na}}(\Lambda_\phi) \sqsubset \mathcal{A}_{\text{na}}(\Lambda_\psi)$  if and only if the set of conjugacy classes carried by  $\mathcal{A}_{\text{na}}(\Lambda_\phi)$  is a subset of the set of conjugacy classes carried by  $\mathcal{A}_{\text{na}}(\Lambda_\psi)$ ; this is a consequence of [HM13b] Lemma 3.1, which applies because nonattracting subgroup systems are vertex group systems ([HM13c] Proposition 1.4). In the general case the free factor system  $\mathcal{F} = \{[F]\}$  has just one component, the inclusion  $F < F_n$  induces a bijection between  $F$ -conjugacy classes and  $F_n$ -conjugacy classes of elements of  $F$ , and the subgroup systems  $\mathcal{A}_{\text{na}}(\Lambda_\phi \mid \mathcal{F})$  and  $\mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F})$  are vertex group systems in  $F$ , and so inclusion of those two vertex group systems is determined by *inclusion of sets of  $F_n$ -conjugacy classes* of elements of  $F$  that are carried by the  $F_n$ -subgroup systems  $\mathcal{A}_{\text{na}}(\Lambda_\phi)$  and  $\mathcal{A}_{\text{na}}(\Lambda_\psi)$ .

**Lemma 5.11.** *There is a uniform bound to the length  $s$  of a sequence  $\Lambda_{\psi_1} < \Lambda_{\psi_2} < \dots < \Lambda_{\psi_s}$ .*

*Proof.* Denote  $\Lambda_{\psi_i}$  by  $\Lambda_i$ . After passing to a subsequence we may assume that either (1) holds for each term in the sequence or (2) holds each term in the sequence. In the latter case,  $\mathcal{F}(\Lambda_i) = \mathcal{F} = \{[F_r]\}$  is independent of  $i$  and each  $\mathcal{A}_{\text{na}}(\Lambda_{\psi_i} \mid \mathcal{F})$  is a vertex group for  $F_r$  by Proposition 1.4 of [HM13c]. The uniform bound on  $s$  comes from the uniform bound (Proposition 3.2 of [HM13b]) on the length of a strictly decreasing sequence of vertex groups. In the former case the  $\mathcal{F}(\Lambda_i)$ 's are a decreasing sequence of vertex groups for  $F_n$  and we are done for the same reason.  $\square$

## 5.4 Geometric models.

We review further details of “geometric models” and definitions of geometricity for EG strata and for attracting laminations.

Consider a rotationless  $\phi \in \text{Out}(F_n)$ , a representative CT  $f: G \rightarrow G$ , and an EG stratum  $H_r$ . A *weak geometric model* for  $H_r$  consists of a compact connected surface  $S$ , a component  $\partial_0 S$  of  $\partial S$ , a pseudo-Anosov homeomorphism  $\Psi: S \rightarrow S$  taking  $\partial_0 S$  to  $\partial_0 S$ , a 2-complex  $Y$  obtained as a quotient  $j: G_{r-1} \amalg S \rightarrow Y$  by using a gluing map  $\alpha: \partial S - \partial_0 S \rightarrow G_{r-1}$  that is  $\pi_1$ -injective on each component, and an extension of the embedding  $G_{r-1} \hookrightarrow Y$  to an embedding  $G_r \hookrightarrow Y$ , such that the following properties hold: there is a deformation retraction  $d: Y \rightarrow G_r$ ; denoting  $dj = d \circ j: S \rightarrow G_r$ , the maps  $S \xrightarrow{\Psi} S \xrightarrow{dj} G_r$  and



$S \xrightarrow{dj} G_r \xrightarrow{f} G_r$  are homotopic; the intersection of  $G_r$  with  $\partial_0 S$  in  $Y$  is a single point  $x \in H_r - G_{r-1}$ ; and the closed path based at  $x$  that goes around  $\partial_0 S$  is homotopic in  $Y$  to an indivisible Nielsen path in  $G$ . A *geometric model* is obtained as the quotient  $X$  of  $Y \amalg G$  by identifying the copies of  $G_r$  in  $Y$  and in  $G$ ; the deformation retraction extends to  $d: X \rightarrow G$ . The stratum  $H_r$  is *geometric* if a weak geometric model for  $H_r$  exists; if this is the case, then the map  $dj: S \rightarrow G$  is  $\pi_1$ -injective, and if  $\Lambda^u \subset S$  is the unstable geodesic lamination of  $\Psi$  with respect to some hyperbolic structure on  $S$  then the map of line spaces  $\mathcal{B}(\pi_1 S) \rightarrow \mathcal{B}(\pi_1 G) = \mathcal{B}(F_n)$  induced by  $dj$  takes the set of leaves of  $\Lambda^u$  homeomorphically to the attracting lamination  $\Lambda$  corresponding to  $H_r$ . As said earlier, an attracting lamination  $\Lambda \in \mathcal{L}(\phi)$  is *geometric* if the EG stratum corresponding to  $\Lambda$  in *some* representative CT  $\phi$  is geometric, which occurs if and only if this holds for *every* representative CT. See Section 2 for a brief review of conditions on  $H_r$  equivalent to geometricity, and see Section 5.3 of [BFH00] and Section 2 of [HM13b] for more details.

In  $X$  define the *complementary subgraph* to be  $L = (G \setminus H_r) \cup \partial_0 S$ .

**Fact 5.12.** *The inclusion  $L \hookrightarrow X$  is  $\pi_1$ -injective on each component and the image subgroups are mutually malnormal. If  $\mathcal{A}_{na}(\Lambda)$  fills then the following hold:*

- (i)  $\mathcal{A}_{na}(\Lambda)$  is represented in  $X$  by the subgraph  $L$ , meaning  $\mathcal{A}_{na}(\Lambda)$  equals the set of conjugacy classes of the image subgroups.
- (ii)  $L$  is disjoint from the manifold interior of  $S$ , and the latter is therefore an open subset of  $X$ .

*Proof.* The first statement follows from Lemma 2.7 of [HM13b]. To prove (i), by [HM13c] Definition 1.2 and Remark 1.3 it follows that  $\mathcal{A}_{na}(\Lambda_m)$  is represented by a subgraph  $K \subset L$  that contains  $G_{r-1} \cup \partial_0 S$ , and by the assumption that  $\mathcal{A}_{na}(\Lambda)$  fills it follows that the subgraph  $K$  must also contain every edge of  $G \setminus G_r$ , and so  $K = L$ . The proof of (ii) follows from [HM13b] Definition 2.10, which shows that at any “attaching point” where  $L$  touches the interior of  $S$ , one can pull  $L$  away from  $S$  inserting an edge, constructing a topological model which exhibits  $L$  as being supported on a proper free factor.  $\square$

When  $\Lambda \in \mathcal{L}(\phi)$  is geometric, and when a geometric model is specified as above, the surface  $S$  together with the associated monomorphism  $\mu = dj_*: \pi_1(S) \hookrightarrow F_n$  will be called the *surface system associated to  $\Lambda$  with respect to the geometric model for  $\phi$* . The following fact says, in essence, that the surface system associated to  $\Lambda$  is well-defined independent of the choice of geometric model.

**Fact 5.13.** *Consider  $\phi \in \text{Out}(F_n)$  and a geometric  $\Lambda \in \mathcal{L}(\phi)$  with the property that  $\mathcal{A}_{na}(\Lambda)$  fills  $F_n$ . Let  $\mu^i: \pi_1(S^i) \hookrightarrow F_n$  ( $i = 1, 2$ ) be the two surface systems associated to  $\Lambda$  with respect to geometric models of EG strata corresponding to  $\Lambda$  in two CTs representing rotationless iterates of  $\phi$ . Then there exists a homeomorphism  $h: S^1 \rightarrow S^2$ , unique up to isotopy, and an inner automorphism  $i: F_n \rightarrow F_n$ , such that  $i \circ \mu^1 = \mu^2 \circ h_*: \pi_1(S^1) \rightarrow F_n$ .*

*Proof.* To prove isotopy uniqueness of  $h$ , given another such  $h': S^1 \rightarrow S^2$ , the two monomorphisms  $\mu_*^2 \circ h_*$ ,  $\mu_*^2 \circ h'_*: \pi_1(S^1) \rightarrow F_n$  differ by postcomposing with an inner automorphism of  $F_n$ , and so the two monomorphisms  $\mu_*^2$ ,  $\mu_*^2 \circ h'_* \circ h_*^{-1}: \pi_1(S^2) \rightarrow F_n$  differ by postcomposing with inner automorphism of  $F_n$ . Since the image of  $\pi_1(S^2)$  in  $F_n$  is its own normalizer

([HM13c] Lemma 2.7 (2)), the latter two monomorphisms differ by precomposition with an inner automorphism of  $\pi_1(S^2)$ . It follows that  $h'_* \circ h_*^{-1}$  is itself an inner automorphism of  $\pi_1(S^2)$ . By the Dehn-Nielsen-Baer theorem [FM12],  $h'h^{-1}$  is isotopic to the identity.

We turn to proof of existence of  $h$ . For  $i = 1, 2$  let  $f^i: G^i \rightarrow G^i$  be CTs representing a rotationless power of  $\phi$  with EG strata  $H_{r_i}^i$  associated to  $\Lambda$ , and let  $X^i$  be a geometric model for  $H_{r_i}^i$  with all accompanying notations indicated with a superscript  $i$ , so that  $\mu^i = dj^i: \pi_1(S^i) \rightarrow \pi_1(G^i) \approx F_n$  is the associated surface system. From any marking change map  $g: G^1 \rightarrow G^2$  we obtain a marking change map  $X^1 \xrightarrow{d^1} G^1 \xrightarrow{g} G^2 \subset X^2$  also denoted  $g: X^1 \rightarrow X^2$ . Let  $\rho^i$  be the indivisible Nielsen path in  $G^i$  of height  $r_i$ , with base point  $x^i = H_{r_i}^i \cap \partial_0 S^i$  and let  $L^i = (G^i \setminus H_{r_i}^i) \cup \partial_0 S^i$  be the complementary subgraph.

Since  $[L^1] = \mathcal{A}_{\text{na}}(\Lambda) = [L^2]$ , by the homotopy extension theorem we may homotope the marking change map  $g: X^1 \rightarrow X^2$  so as to take  $L^1$  to  $L^2$  by a homotopy equivalence. By Fact 5.12 we may apply the following fact to the homotopy equivalence of pairs  $g: (X^1, L^1) \rightarrow (X^2, L^2)$ :

**Fact 5.14.** *For  $i = 1, 2$  let  $K^i$  be a finite graph,  $S^i$  a compact connected surface with nonempty boundary and negative Euler characteristic,  $\beta^i: \partial S^i \rightarrow K^i$  a map that is  $\pi_1$ -injective on each component, and  $j^i: K^i \amalg S^i \rightarrow X^i$  the quotient map to a finite connected complex defined by identifying  $x \sim \beta^i(x)$  for each  $x \in \partial S^i$ . For any homotopy equivalence of pairs  $g: (X^1, K^1) \rightarrow (X^2, K^2)$  there exists a homeomorphism  $h: S^1 \rightarrow S^2$  such that the maps  $j^2 \circ h, g \circ j^1$  are homotopic.*

Putting off its proof for a moment, from the conclusion of Fact 5.14 it follows that the maps  $dj^2 \circ h, g \circ dj^1: S^1 \rightarrow G^2$  are homotopic, and so the two monomorphisms the two monomorphisms  $dj_*^2 \circ h_*, dj_*^1 \circ g_* = dj_*^1: \pi_1 S^1 \rightarrow \pi_1 G^2 = F_n$  differ by postcomposing with an inner automorphism of  $F_n$ . This completes the proof of Fact 5.13.  $\square$

*Proof of Fact 5.14.* This proof is very close to that of [HM13b] Lemma 2.21, which is the special case that equations  $X^1 = X^2$  and  $K^1 = K^2$  hold, but one major step is different. Denote  $g^1 = g$ , which we may assume is simplicial, by the simplicial approximation theorem. Following [SW79], decompose  $X^1$  into a graph of spaces: the edge spaces are the interiors of the components of a regular neighborhood  $N(\partial S^1)$ ; one vertex space is  $\text{cl}(S^1 - N(\partial S^1))$  whose fundamental group is identified with  $\pi_1(S^1)$ ; the other vertex spaces are the components of  $K^1$ . By a “curve” we mean a homotopically nontrivial closed curve. Applying Bass-Serre theory to this graph of spaces, and using that curves in distinct components of  $\partial S^1$  are not homotopic in  $S^1$ , one concludes the following: for each curve  $c$  in  $S^1$ , if  $c$  is homotopic in  $X^1$  to a curve  $c'$  in  $S^1$  such that  $c, c'$  are not homotopic in  $S^1$ , or if  $c$  is homotopic in  $X^1$  to a curve in  $K^1$ , then  $c$  is homotopic into  $\partial S^1$ ; and if two curves in  $K^1$  are homotopic in  $X^1$  then they are homotopic in  $K^1$ .

Consider the subcomplex  $Z^1 = (g^1)^{-1}(K^2) \subset X^1$  which contains  $K^1$ . If  $c$  is a nonperipheral curve in  $\text{int}(S^1)$  then  $c$  is not contained in  $Z^1$  because if it were then there would be a curve  $c'$  in  $K^1$  such that  $g^1(c), g^1(c')$  are homotopic, forcing  $c, c'$  to be homotopic, a contradiction. Each component of  $Z^1$  is therefore contained either in a neighborhood of a component of  $\partial S^1$  or in a subdisc of  $S^1$ . Components of  $Z^1$  of the latter type can be removed by homotopy. It follows that, with appropriate choice of base points,  $(g^1)_*(\pi_1 S^1) \subset \pi_1 S^2$ . Using a homotopy inverse  $g^2: (X^2, K^2) \rightarrow (X^1, K^1)$ , the exact same argument applies with

the superscripts 1, 2 reversed. It follows in turn that  $(g^1)_*(\pi_1 S^1) = \pi_1 S^2$  (the corresponding step of [HM13b] Lemma 2.21 applies [BFH00] Lemma 6.0.6 which does not apply here). This isomorphism takes peripheral elements of  $\pi_1 S^1$  to peripheral elements of  $\pi_1 S^2$ , and so by the Dehn-Nielsen-Baer theorem it is induced by a homeomorphism  $h: S^1 \rightarrow S^2$ , which evidently satisfies the conclusions of the lemma.  $\square$

**Fact 5.15.** *Suppose that  $S$  is a surface with boundary and that  $\mu \in \text{MCG}(S)$  corresponds to  $\phi \in \text{Out}(F_n)$  under an identification of  $\pi_1(S)$  with  $F_n$ . Then  $\mathcal{L}(\phi)$  is the set of unstable laminations for a Thurston decomposition of  $\mu$ .*

*Proof.* After passing to an iterate of  $\mu$  we may assume that each unstable lamination  $\Lambda_\mu$  for  $\mu$  has a  $\mu$ -invariant leaf  $\gamma_\mu$ . That leaf is birecurrent and non-periodic because  $\Lambda_\mu$  is minimal and not a closed curve. It is well known (see for example Proposition 2.1 of [HM13b]) that  $\gamma_\mu$  has an attracting neighborhood in the weak topology for some iterate of  $\mu$  so  $\Lambda_\mu \in \mathcal{L}(\phi)$  by Definition 3.1.5 of [BFH00].

Suppose now that  $\Lambda_\phi \in \mathcal{L}(\phi)$ . Choose a hyperbolic structure on  $S$  in which the reducing curves in the Thurston decomposition of  $\mu$  are geodesic. After passing to an iterate we may assume that components in the Thurston decomposition are invariant, that the restriction of  $\mu$  to each component is either the identity or pseudo-Anosov, that  $\phi$  is rotationless and that there is a  $\mu$ -invariant leaf  $\gamma_\phi$  in the realization of  $\Lambda_\phi$  in  $S$ . Suppose that  $[a] \in F_n$  is weakly attracted to  $\Lambda_\phi$  under the action of  $\phi$ . For each  $k \geq 0$ , let  $\alpha_k$  be the closed geodesic in  $S$  corresponding to  $\phi^k([a]) = \mu^k([\alpha_0])$ . The number of intersections of  $\alpha_k$  with the set of reducing curves of the Thurston decomposition is independent of  $k$  while the length of  $\alpha_k$  goes to  $\infty$ . It follows that the ends of the geodesic in  $S$  corresponding to any weak limit of the  $\alpha_k$ 's are disjoint from the reducing curves. Applying this to the birecurrent weak limit  $\gamma_\phi$  we have that  $\gamma_\phi$  is realized by a geodesic that is disjoint from the reducing curves and so contained in a single, necessarily pseudo-Anosov, component  $S_0$  of the Thurston decomposition. We may assume without loss that  $\alpha \subset S_0$ .

Letting  $\Lambda_0^u$  and  $\Lambda_0^s$  be the unstable and stable foliations for  $\mu|_{S_0}$ , we may homotop  $\alpha_0$  to a closed curve  $\alpha'_0$  that it is an alternating concatenation of geodesic paths  $\sigma_i \subset \Lambda_0^u$  and  $\tau_i \subset \Lambda_0^s$ . By iterating the pseudo-Anosov homeomorphism representing  $\mu|_{S_0}$ , we obtain closed curves  $\alpha'_k$  realizing  $\phi^k([a])$  that decompose into a concatenation of geodesic paths in  $\Lambda_0^u$  whose lengths  $\rightarrow \infty$  and paths in  $\Lambda_0^s$  whose lengths  $\rightarrow 0$ . It follows that any birecurrent weak limit of the  $\alpha'_k$ 's is a weak limit of  $\Lambda_0^u$ , and so by minimality, is a leaf of  $\Lambda_0^u$ . Thus  $\gamma_\phi$  is dense in both  $\Lambda_0^u$  and  $\Lambda_\phi$  so these two laminations are equal and we are done.  $\square$

Associated to a compact surface  $S$  with nonempty boundary are its mapping class group  $\text{MCG}(S)$ , the group of homeomorphisms of  $S$  modulo isotopy, and its boundary relative mapping class group  $\text{MCG}(S, \partial S)$ , the group of all homeomorphisms that restrict to the identity on  $\partial S$  modulo isotopy through such homeomorphisms. We denote the finite index subgroup of  $\text{MCG}(S)$  consisting of elements that setwise fix each component of  $\partial S$  by  $\text{MCG}_0(S)$ . The induced surjective homomorphism  $\text{MCG}(S, \partial S) \rightarrow \text{MCG}_0(S)$  will be called the *despinning homomorphism*.

Given  $\Lambda_\phi \in \mathcal{L}(F_n)$  with associated surface system  $\pi_1 S \hookrightarrow F_n$ , we identify  $\pi_1 S$  as a subgroup  $\pi_1 S < F_n$  with conjugacy class  $[\pi_1 S]$  and stabilizer subgroup  $\text{Stab}[\pi_1 S] < \text{Out}(F_n)$ . The subgroup  $\pi_1 S$  is its own normalizer in  $F_n$  ([HM13b] Lemma 2.7 (2)), and so there is a

well-defined induced homomorphism  $\text{Stab}[\pi_1 S] \rightarrow \text{Out}(\pi_1 S)$  ([HM13b] Fact 1.4). Associated to each oriented component of  $\partial S$  is a conjugacy class in the group  $\pi_1 S$  called a *peripheral conjugacy class*. According to the Dehn-Nielsen-Baer Theorem the subgroup of  $\text{Out}(\pi_1 S)$  that preserves the set of peripheral conjugacy classes is naturally isomorphic to  $\text{MCG}(S)$ .

**Example 5.16.** Suppose that  $F_{m+2} = F_m * \langle A, B \rangle$  and that  $s \in F_m$  fills  $F_m$ . In Example 4.1, we constructed an element  $\phi \in \text{Out}(F_{m+2})$  with a filling lamination  $\Lambda$  whose stabilizer contains every element  $\theta \in \text{Out}(F_{m+2})$  that is represented by an automorphism  $\Theta$  that fixes each element of  $\langle A, B, s \rangle$ . In this example we apply this to produce an example in which the stabilizer of  $\Lambda$  contains a boundary relative mapping class group.

Let  $X$  be the two complex obtained by attaching a pair of loops  $A, B$  to a basepoint  $v \in \sigma$ . Identify  $\pi_1(S, v)$  with some  $F_m$  and  $\pi_1(X, v)$  with  $F_{m+2} = F_m * \langle A, B \rangle$ ; let  $s \in F_m$  be the element determined by  $\sigma$ . Each  $\nu \in \text{MCG}(S, \partial S)$  is represented by a homeomorphism  $h : S \rightarrow S$  that pointwise fixes  $\sigma$  and so extends by the identity on  $A$  and  $B$  to a homotopy equivalence of  $X$  whose induced action on  $\pi_1(X, v)$  fixes  $\langle \sigma, A, B \rangle$ . There is an induced injective homomorphism  $\text{MCG}(S, \partial S) \rightarrow \text{Out}(F_{m+2})$  whose image we denote  $\mathcal{H}$ . Since  $s$  fills  $F_m$  (Lemma 2.5 of [HM13b] (1)),  $\mathcal{H} \cap \text{IA}_n(\mathbb{Z}/3) < K$ .

**Lemma 5.17.** *Suppose that  $\mathcal{H} < K$ , that  $\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_k = \{[F_n]\}$  is a maximal  $\mathcal{H}$ -invariant filtration by properly nested free factor systems and that  $\mathcal{F}_{n_g}$  is one of the  $\mathcal{F}_i$ 's. Let  $1 \leq i_1 < \cdots < i_M \leq k$  be the indices  $i$  for which  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  is a multi-edge extension. Suppose also that a rotationless  $\phi_m \in \mathcal{H}$  is given for each  $m = 1, \dots, M$  so that  $\phi_m$  is irreducible rel  $\mathcal{F}_{i_m-1} \sqsubset \mathcal{F}_{i_m}$  with associated attracting geometric lamination  $\Lambda_m$ , and let  $\pi_1 S_m \hookrightarrow F_n$  be the surface system associated to  $\Lambda_m \in \mathcal{L}(F_n)$ . Then*

- (1)  $\mathcal{H}$  stabilizes each  $[\pi_1 S_m]$ , and the induced homomorphism  $\mathcal{H} \mapsto \text{Stab}[\pi_1 S_m] \hookrightarrow \text{Out}(\pi_1 S_m)$  has image in  $\text{MCG}(S_m)$ , inducing a homomorphism  $\xi_m : \mathcal{H} \rightarrow \text{MCG}(S_m)$ .
- (2) For each  $\psi \in \mathcal{H}$  and  $\Lambda_\psi \in \mathcal{L}(\psi)$  there exists  $m \in \{1, \dots, M\}$  such that the following hold:
  - (a)  $\Lambda_\psi$  is carried by  $[\pi_1 S_m]$ .
  - (b) Either  $\Lambda_\psi \prec \Lambda_m$  or  $\psi$  is irreducible rel  $\mathcal{F}_{i_m-1} \sqsubset \mathcal{F}_{i_m}$ .
  - (c)  $\xi_m(\psi)$  is non-trivial.
- (3) There is a homomorphism  $\Theta : \prod_1^M \text{MCG}(S_m, \partial S_m) \rightarrow \text{Ker}(PF) \cap \bigcap_1^M \text{Stab}[\pi_1 S_m]$  such that its composition with the homomorphism  $\bigcap_1^M \text{Stab}[\pi_1 S_m] \rightarrow \prod_1^M \text{Out}(\pi_1 S_m)$  has image in  $\prod_1^M \text{MCG}(S_m)$ , and the composition  $\prod_1^M \text{MCG}(S_m, \partial S_m) \rightarrow \prod_1^M \text{MCG}(S_m)$  is the product of the despinning homomorphisms and in particular has finite index image.

*Proof.* Throughout this proof, by applying the existence theorem (Theorem 4.28 of [FH11]) for CTs, we choose the CTs representing all rotationless elements of  $\mathcal{H}$  to have core filtration elements representing the free factor systems  $\mathcal{F}_1 \sqsubset \cdots \sqsubset \mathcal{F}_k$ .

For the proofs of (1), (2) we set the notation of appropriate geometric models. For each  $m = 1, \dots, M$  choose a CT  $f^m : G^m \rightarrow G^m$  representing  $\phi_m$ . Let  $H_{r_m}^m \subset G^m$  be the EG geometric stratum corresponding to  $\Lambda_m$ . Let  $X_m$  be a geometric model for  $H_{r_m}^m$ , and

$Y_m \subset X_m$  a weak geometric model, and let  $S_m$  be the surface of the geometric model with upper boundary  $\partial_0 S_m$ ; let  $\partial_- S_m = \partial S_m - \partial_0 S_m$  denote the *lower boundary*. Thus, there is a quotient map  $j_m: G_{r_m-1}^m \amalg S_m \rightarrow Y_m$  defined by attaching each component of  $\partial_- S_m$  to  $G_{r_m-1}^m$  by a  $\pi_1$ -injective map, the embedding  $G_{r_m-1}^m \hookrightarrow Y_m$  extends to an embedding  $G_{r_m}^m \rightarrow Y_m$ , and  $X_m$  is the quotient of  $Y_m \amalg G^m$  by identifying the copies of  $G_{r_m}^m$ . Let  $L_m = (G^m \setminus H_{r_m}^m) \cup \partial_0 S_m$  be the complementary subgraph.

Conclusion (1) exactly matches the conclusion of Theorem J of [HM13d], although that theorem is stated only for the case when the geometric stratum in question is the top stratum. But we easily reduce to that case by restricting  $\phi_m$  to the component  $\mathcal{F}' = \{[F']\}$  of  $\mathcal{F}_{i_m}$  that carries  $\Lambda_m$ : the restriction  $\phi' = \phi_m|_{F'} \in \text{Out}(F')$  is rotationless; it is represented by the CT  $f': G' \rightarrow G'$  which is the restriction of  $f^m$  to the component  $G^m$  containing  $H' = H_{r_m}^m$ , and  $H'$  is its top stratum; the component  $Y'$  of  $Y_m$  that contains  $H'$  is a geometric model for  $H'$  relative to  $f'$ ; and  $S_m$  is the surface of that geometric model. Theorem J then indeed applies in that restricted context, and its conclusions immediately imply item (1).

For the proof of conclusion (2), fix  $\psi \in \mathcal{H}$  and  $\Lambda_\psi \in \mathcal{L}(\psi)$ . Passing to a power we may assume  $\psi$  is rotationless. Choose a CT representing  $\psi$  with core filtration elements  $G_{r(i)}$  representing each  $\mathcal{F}_i$ . Letting  $r(0) = 0$  and letting  $H_s$  be the stratum corresponding to  $\Lambda_\psi$ , choose  $i$  so that  $r(i-1) < s \leq r(i)$ . Then  $\Lambda_\psi$  is carried by  $\mathcal{F}_i$  but not by  $\mathcal{F}_{i-1}$  and hence  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  is a multi-edge extension. In particular,  $i = i_m$  for some  $m = 1, \dots, M$ .

Applying (1),  $\psi$  preserves  $[\pi_1 S_m]$  and restricts to a mapping class  $\psi_m = \xi_m(\psi) \in \text{MCG}(S_m) < \text{Out}(\pi_1 S_m)$ . As in the proof of (1), we may assume that  $S_m$  corresponds to the highest stratum of  $f^m: G^m \rightarrow G^m$  or equivalently that  $X_m = Y_m$ . Let  $N_- \subset S_m$  be a collar neighborhood of  $\partial_- S_m$  in  $Y_m$  and let  $S_m^* = \text{cl}(S_m - N_-)$ . The inclusion map  $S_m^* \hookrightarrow S_m$  is homotopic to a homeomorphism, inducing an isomorphism  $\text{MCG}(S_m^*) \approx \text{MCG}(S_m)$ . The quotient map  $j$  restricts to an embedding  $S_m^* \hookrightarrow Y_m$ , and  $\text{cl}(Y_m - S_m^*) = G_{r_m-1}^m \cup j(N_-)$  deformation restricts to  $G_{r_m-1}^m$ . As in the proof of Fact 5.13, the homotopy extension theorem and Fact 5.14 implies the existence of a homotopy equivalence  $\Psi: Y_m \rightarrow Y_m$  representing  $\psi$  such that  $\Psi$  restricts to a homeomorphism of  $S_m^*$  and to a self-homotopy equivalence of  $G_{r_m-1}^m \cup j(N_-)$ .

Our strategy for proving (2a) is to choose a closed curve  $\gamma$  representing a conjugacy class  $[\gamma]$  that is weakly attracted to  $\Lambda_\psi$  under iteration by  $\Psi$  and prove that each birecurrent non-periodic weak limit of  $\psi$ -iterates of  $[\gamma]$  is either contained in  $S_m$  or contained in  $G_{r_m-1}^m$ . Since a generic leaf of  $\Lambda_\psi$  can not satisfy the latter it must satisfy the former and we will be done. To carry out this strategy we have to understand how  $\Psi$ -iterates of  $\gamma$  intersect  $\partial_- S_m^*$ .

Pick a regular neighborhood  $N \subset S_m^*$  of  $\partial_- S_m^*$ , with associated fibration  $N \mapsto \partial_- S_m^*$  whose fibers are compact arcs. We obtain a graph of spaces decomposition of  $Y_m$  in the sense of [SW79], having as edge spaces the components of  $\text{int}(N)$ , and as vertex spaces the components of  $Y_m - \text{int}(N)$ , including the “surface” vertex space  $\widehat{S}_m = \text{cl}(S_m^* - N)$ . By applying Bass-Serre theory [SW79], the only conjugacy classes carried by both the surface vertex group and some non-surface vertex group are those carried by some edge group, and the same statement follows immediately for lines. A closed curve  $\gamma$  in  $Y_m$  is *efficient* if  $\gamma \subset \text{int}(N)$ , or if  $\gamma \cap \text{int}(N) = \emptyset$ , or if  $\gamma$  has an *efficient concatenation*  $\gamma = \alpha_1 \mu_1 \cdots \alpha_{2K} \mu_{2K}$



meaning: each  $\alpha_k$  is a fiber of  $N$ ; each  $\mu_k$  is a *vertex path* meaning a path in a vertex space with endpoints in  $N$  but not homotopic rel endpoints into  $N$ ; and the  $\mu_k$  alternate between surface vertex paths and non-surface vertex paths. By applying Bass-Serre theory [SW79] it follows that if  $\gamma, \gamma'$  are homotopic efficient curves in  $Y_m$  then they are of the same type: both are homotopic into  $N$ ; or neither is homotopic into  $N$  but either both are homotopic into  $\text{cl}(Y_m - \widehat{S}_m)$  or both into  $\text{int}(\widehat{S}_m)$ ; or both have efficient concatenations of the same length  $2K$  and, after a cyclic permutation, the same sequence of vertex paths  $\mu_k$  up to homotopy through vertex paths. Note that we may isotope  $\Psi$  so that, besides preserving  $S_m^*$ , it also preserves  $N$ , the fibers of  $N$  and  $\widehat{S}_m$ .

Suppose now that the efficient closed curve  $\gamma$  is weakly attracted to  $\Lambda_\psi$ . If  $\gamma$  is carried by  $G_{r_m-1}^m$  then so are all of its  $\Psi$ -iterates and their weak limits, contradicting that  $\Lambda_\psi$  is not carried by  $G_{r_m-1}^m$ ; so  $\gamma$  must intersect  $S_m^*$ . If  $\gamma \subset S_m^*$  then all of its  $\Psi$ -iterates are contained in  $S_m^*$  and we are done. Suppose then that  $\gamma$  has an efficient concatenation  $\gamma = \alpha_1 \mu_1 \cdots \alpha_{2K} \mu_{2K}$  and hence that  $\Psi^i(\gamma) = \Psi^i(\alpha_1) \Psi^i(\mu_1) \cdots \Psi^i(\alpha_{2K}) \Psi^i(\mu_{2K})$  is an efficient concatenation for each  $i \geq 1$ . Since the length of  $[\psi^i(\gamma)]$  goes to infinity, each weak limit  $L$  of the  $[\psi^i(\gamma)]$ 's has an efficient concatenation with at most  $2K$  terms. In particular, the ends of  $L$  are rays in one of the vertex spaces. Taking  $L$  to be a generic leaf of  $\Lambda_\psi$ , since  $L$  is birecurrent then  $L$  itself is contained in one of the vertex spaces and the proof of (2a) is complete. Fact 5.15 implies that  $\Lambda_\psi$  is the unstable foliation for a pseudo-Anosov component  $S' \subset S_m$  in the Thurston decomposition for  $\xi_m(\psi)$ . In particular, (2c) is satisfied.

For (2b) let  $\mathcal{C}(S_m)$  be the set of conjugacy classes carried by  $[\pi_1(S_m)]$ . The ‘span argument’ for geometric strata (see Lemma 7.0.7 of [BFH00] or Proposition 2.15(4) of [HM13b]) implies that  $\mathcal{F}(\Lambda_m)$  carries each element of  $\mathcal{C}(S_m)$ . Since  $\Lambda_\psi$  is a weak limit of elements of  $\mathcal{C}(S_m)$ , it is carried by  $\mathcal{F}(\Lambda_m)$  proving that  $\mathcal{F}(\Lambda_\psi) \subset \mathcal{F}(\Lambda_m)$ . If  $\mathcal{F}(\Lambda_\psi) \neq \mathcal{F}(\Lambda_m)$  then  $\Lambda_\psi \prec \Lambda_m$  and we are done so suppose that  $\mathcal{F}(\Lambda_\psi) = \mathcal{F}(\Lambda_m)$ , which we now simply call  $\mathcal{F}$ .

A conjugacy class is carried by  $\mathcal{A}_{\text{na}}(\Lambda_{\phi_m} \mid \mathcal{F})$  if and only if it is carried by  $\mathcal{A}_{\text{na}}(\Lambda_m)$  and by  $\mathcal{F}$ . We know that  $\mathcal{F} \subset \mathcal{F}_{i_m}$  and that a conjugacy class in  $\mathcal{F}_{i_m}$  is carried by  $\mathcal{A}_{\text{na}}(\Lambda_m)$  if and only if it is carried by  $\mathcal{F}_{i_m-1}$  or is represented by an iterate of  $\partial_0 S_m$ . It follows that a conjugacy class is carried by  $\mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F})$  if and only if it is carried by  $\mathcal{F} \wedge \mathcal{F}_{i_m-1}$  or is represented by an iterate of  $\partial_0 S_m$ . Each such conjugacy class is carried by  $\mathcal{A}_{\text{na}}(\Lambda_\psi)$  so  $\mathcal{A}_{\text{na}}(\Lambda_{\phi_m} \mid \mathcal{F}) \subset \mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F})$ . If  $\mathcal{A}_{\text{na}}(\Lambda_{\phi_m} \mid \mathcal{F}) \neq \mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F})$  then  $\Lambda_\psi \prec \Lambda_m$  and we are done so suppose that  $\mathcal{A}_{\text{na}}(\Lambda_{\phi_m} \mid \mathcal{F}) = \mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F})$ . In this case every non-peripheral element of  $[\pi_1(S_m)]$  is weakly attracted to  $\Lambda_\psi$  so  $\xi_m(\psi)$  is a pseudo-Anosov mapping class and to complete the proof of (2) it remains to show that  $\psi$  is irreducible relative to  $F_{i_m-1} \subset F_{i_m}$ .

A conjugacy class  $[a]$  carried by  $\mathcal{F}_{i_m}$  but not by  $\mathcal{A}_{\text{na}}(\Lambda_{\phi_m})$  is represented by a closed curve  $\gamma$  that is either contained in  $S_m$  or has an efficient representation in  $Y_m$  with at least one term  $\mu_j$  that is a surface vertex path. By Nielsen–Thurston theory (specifically [HM13b] Proposition 2.14), either  $\gamma$  is homotopic to a power of  $\partial_0 S_m$  or  $\psi^i([a])$  weakly converges to  $\Lambda_\psi$ . This proves that  $\mathcal{A}_{\text{na}}(\Lambda_\psi \mid \mathcal{F}_{i_m}) = \mathcal{F}(G_{i_m-1}) \cup \langle \partial_0 S_m \rangle$ .

If  $\mathcal{F}' \subset \mathcal{F}_{i_m}$  is a  $\psi$ -invariant free factor system that properly contains  $\mathcal{F}_{i_m-1}$  then  $\mathcal{F}'$  contains a conjugacy that is not carried by  $\mathcal{A}_{\text{na}}(\Lambda_\psi)$  and so  $\mathcal{F}'$  also carries  $\Lambda_\psi$ . Lemma 7.0.7 and Corollary 7.0.8 of [BFH00] therefore imply that  $\mathcal{F}'$  carries  $[\pi_1(S_m)]$  and hence also  $\Lambda_m$ .



Since  $\phi$  is irreducible rel  $\mathcal{F}_{i_m-1} \sqsubset \mathcal{F}_{i_m}$ , it must be that  $\mathcal{F}' = \mathcal{F}_{i_m}$  and so  $\psi$  is irreducible rel  $\mathcal{F}_{i_m-1} \sqsubset \mathcal{F}_{i_m}$ . This completes the proof of (2).

The first step in proving (3) is to show that if  $m \neq m'$  then  $[\pi_1 S_{m'}]$  is carried by  $\mathcal{A}_{\text{na}}(\Lambda_m)$ . This is obvious for  $m' < m$  because  $[\pi_1 S_{m'}]$  is carried by  $\mathcal{F}_{i_{m'}} \sqsubset \mathcal{F}_{i_{m-1}} \sqsubset \mathcal{A}_{\text{na}}(\Lambda_m)$ . On the other hand, if  $m' > m$  then it follows from the geometric model for  $\phi_{m'}$  that  $\Lambda_m$  is not carried by  $[\pi_1 S_{m'}]$ . It then follows from the  $\phi_m$ -invariance of  $[\pi_1 S_{m'}]$  that no conjugacy class in  $[\pi_1 S_{m'}]$  is weakly attracted to  $\Lambda_m$  and so each conjugacy class in  $[\pi_1 S_{m'}]$  is carried by  $\mathcal{A}_{\text{na}}(\Lambda_m)$  as desired.

For each  $m \in \{1, \dots, M\}$  and each  $\nu \in \text{MCG}(S_m, \partial S_m)$  let  $g(m, \nu) : X_m \rightarrow X_m$  be a homotopy equivalence that restricts to the identity on  $L_m$  and to a homeomorphism that represents  $\nu$  on  $S_m$ ; let  $\theta(m, \nu)$  be the element of  $\text{Out}(F_n)$  determined by  $g(m, \nu)$ . Since  $\mathcal{A}_{\text{na}}(\Lambda_m)$  is realized by  $L_m$ , it follows that  $\theta(m, \nu)$  fixes each leaf of  $\Lambda_\eta^+$  (Lemma 5.1) and fixes each conjugacy class carried by  $[\pi_1 S_{m'}]$  for  $m \neq m'$ . The former implies that  $\theta(m, \nu) \in \text{Ker}(PF)$  and the latter implies that  $\theta(m, \nu)$  preserves  $[\pi_1 S_{m'}]$  and induces the identity element of  $\text{MCG}(S_{m'})$  for each  $m' \neq m$ . By construction  $\theta(m, \nu)$  preserves  $[\pi_1 S_m]$  and induces the element of  $\text{MCG}(S_m)$  that is the image of  $\nu$  under the despinning homomorphism.

Define  $\Theta : \prod_1^M \text{MCG}(S_m, \partial S_m) \rightarrow \text{Ker}(PF) \cap \bigcap_1^M \text{Stab}[\pi_1 S_m]$  by

$$\Theta(\nu_1, \dots, \nu_M) = \theta(1, \nu_1) \circ \theta(2, \nu_2) \circ \dots \circ \theta(M, \nu_M)$$

The restriction of  $\Theta(\nu_1, \dots, \nu_M)$  to each  $[\pi_1(S_m)]$  is a well defined element of  $\text{MCG}(S_m)$  so we have an induced homomorphism

$$\prod_1^M \text{MCG}(S_m, \partial S_m) \rightarrow \prod_1^M \text{MCG}(S_m)$$

which by construction satisfies

$$\Theta(\nu_1, \dots, \nu_M) \mapsto (\nu'_1, \dots, \nu'_M)$$

where  $\nu'_m$  is the image of  $\nu_m$  under the despinning homomorphism. This completes the proof of (3) and also the proof of lemma.  $\square$

*Proof of Theorem 1.4.* Enumerate  $K = \{\psi_a \mid a = 1, 2, \dots\}$  and define a nested sequence of finitely generated subgroups  $A^1 < A^2 < A^3 < \dots$  with  $A^a = \langle \psi_1, \dots, \psi_a \rangle$ . Note that  $K = \cup A^a$  is finitely generated if and only if  $A^a = K$  for some  $a$ .

Consider one of the finitely generated subgroups  $A = A^a$ . Choose  $\emptyset = \mathcal{F}_0 \sqsubset \mathcal{F}_1 \sqsubset \dots \sqsubset \mathcal{F}_I = \{[F_n]\}$  to be a maximal  $A$ -invariant filtration by free factor systems, one of which is  $\mathcal{F}_{\text{ng}}$ . Let  $1 \leq i_1 < \dots < i_M \leq I$  be the indices  $i$  for which  $\mathcal{F}_{i-1} \sqsubset \mathcal{F}_i$  is a multi-edge extension. By Theorem D of [HM13a] for each  $m = 1, \dots, M$  there exist  $\phi_m \in A$  such that  $\phi_m$  is irreducible rel  $\mathcal{F}_{i_m-1} \sqsubset \mathcal{F}_{i_m}$ ; let  $\Lambda_{\phi_m}$  be the corresponding geometric (by Lemma 5.1) element of  $\mathcal{L}(\phi_m)$ , and let  $\pi_1(S_m) \hookrightarrow F_n$  be the associated surface system. Applying Lemma 5.17 to  $\mathcal{H} = A$  using  $\phi_1, \dots, \phi_M \in A$ , for each  $m = 1, \dots, M$  let  $\xi_m : A \rightarrow \text{MCG}(S_m)$  be the homomorphism given by Lemma 5.17 (1). Define  $\xi = (\xi_1, \dots, \xi_M) : A \rightarrow \text{MCG}(S_1) \times$

$\cdots \times \text{MCG}(S_M)$ . Item Lemma 5.17 (2c) implies that  $\text{Ker}(\xi)$  is UPG, and Lemma 5.5 then implies that  $\text{Ker}(\xi)$  is linear, abelian and finitely generated.

For  $m = 1, \dots, M$  let  $p_m$  be the maximal length of a chain  $\Lambda_{\theta_1} \prec \Lambda_{\theta_2} \prec \dots \prec \Lambda_{\phi_m}$  for  $\Lambda_{\theta_1}, \Lambda_{\theta_2}, \dots \in \mathcal{L}(F_n)$ . By Lemma 5.11 we have  $p_m \leq P$  for some fixed  $P$ . Define the complexity  $cx = cx^a$  to be the sequence of  $p_m$ 's in non-decreasing order. Since  $M \leq I \leq 2n - 1$ , the complexity  $cx$  is an element of a finite set depending only on  $n$  and  $P$ , namely the nondecreasing sequences of length  $\leq 2n - 1$  with entries in  $\{1, 2, \dots, P\}$ . Order this set lexicographically.

Consider the next finitely generated subgroup  $A' = A^{a+1} = \langle \psi_1, \dots, \psi_a, \psi_{a+1} \rangle$ . Choose  $\emptyset = \mathcal{F}'_0 \subset \mathcal{F}'_1 \subset \dots \subset \mathcal{F}'_J = \{[F_n]\}$  to be a maximal  $A'$ -invariant filtration by free factor systems, one of which is  $\mathcal{F}_{\text{ng}}$ , and let  $1 \leq j_1 < \dots < j_L \leq J$  be the indices  $j$  for which  $\mathcal{F}'_{j-1} \subset \mathcal{F}'_j$  is a multi-edge extension. Applying Theorem D of [HM13a] as above, for  $l = 1, \dots, L$  choose  $\phi'_l \in A'$  such that  $\phi'_l$  is irreducible rel  $\mathcal{F}'_{j_l-1} \subset \mathcal{F}'_{j_l}$ , and with the following additional property: if there exists  $m \in \{1, \dots, M\}$  such that  $\phi_m$  is irreducible rel  $\mathcal{F}'_{j_l-1} \subset \mathcal{F}'_{j_l}$  and such that  $\Lambda_{\phi_m}$  is the element of  $\mathcal{L}(\phi_m)$  corresponding to the extension  $\mathcal{F}'_{j_l-1} \subset \mathcal{F}'_{j_l}$  then  $\phi'_l = \phi_m$ . Define  $\Lambda_{\phi'_l}$ ,  $\xi'_l$ ,  $\xi'$ ,  $p'_l$ , and  $cx' = cx^{a+1}$  as above.

What happens in effect is that this procedure ends after a finite number of iterations, although this conclusion only comes after the fact, once we have proved by other means that  $K$  is finitely generated.

Lemma 5.17 (2) (together with choice of  $\phi'_l$  above) implies that each time the construction is repeated as above, for each  $m \in \{1, \dots, M\}$  there exists  $l \in \{1, \dots, L\}$  such that either  $\Lambda_{\phi_m} = \Lambda_{\phi'_l}$  or  $\Lambda_{\phi_m} \prec \Lambda_{\phi'_l}$ ; in the former case  $p_m = p'_l$  and in the latter case  $p_m < p'_l$ . It follows that  $cx$  is less than or equal to  $cx'$  in lexicographical order, with equality holding only if  $M = L$  and for all  $1 \leq m \leq M$  we have  $(\Lambda_{\phi_m}, \phi_m) = (\Lambda_{\phi'_m}, \phi'_m)$ . Since the set of complexities is finite, the complexity sequence  $cx^1, cx^2, cx^3, \dots$  is eventually constant. It follows that the subset of  $\mathcal{L}(F_n)$  given by  $\{(\Lambda_{\phi_m}, \phi_m) \mid 1 \leq m \leq M\}$  is eventually constant, and so by Lemma 5.13 that the surface systems  $\mu_m: \pi_1 S_m \rightarrow F_n$  associated to the  $(\Lambda_{\phi_m}, \phi_m)$  are eventually constant, and hence the group  $\text{MCG}(S_1) \times \dots \times \text{MCG}(S_M)$  is eventually constant. The (eventually defined) sequence of homomorphisms to this group from the groups  $A^1 < A^2 < \dots$  are (eventually) consistent with the inclusions, and hence these homomorphisms fit together to define a homomorphism  $K \mapsto \text{MCG}(S_1) \times \dots \times \text{MCG}(S_M)$  whose kernel is UPG and hence, by Lemma 5.5, is finitely generated, linear and abelian. By Lemma 5.17 (3), the image of this homomorphism has finite index, completing the proof.  $\square$

## 6 Completion of the proof of Theorem 1.2

Let  $\phi, \psi$  have filling lamination pairs  $\Lambda_{\phi}^{\pm}, \Lambda_{\psi}^{\pm}$ . From Corollary 4.23 it follows that if  $\Lambda_{\phi}^+ \neq \Lambda_{\psi}^+$  then  $\partial_- \phi \neq \partial_- \psi$ ; and by inverting  $\phi, \psi$  it follows that if  $\Lambda_{\phi}^- \neq \Lambda_{\psi}^-$  then  $\partial_+ \phi \neq \partial_+ \psi$ .

It remains to show that if  $\Lambda_{\phi}^+ = \Lambda_{\psi}^+ \equiv \Lambda^+$  then  $\partial_+ \phi = \partial_+ \psi$  and  $\partial_- \phi = \partial_- \psi$ . By [BFH00] Corollary 3.3.1 the expansion factor homomorphism has infinite cyclic image, and so after replacing  $\phi$  and  $\psi$  by iterates we may assume that  $\text{PF}_{\Lambda^+}(\phi) = \text{PF}_{\Lambda^+}(\psi)$ . The kernel  $K$  of  $\text{PF}_{\Lambda^+}: \text{Stab}(\Lambda^+) \cap \text{IA}_n(\mathbb{Z}/3) \rightarrow \mathbb{R}$  is finitely generated by Theorem 1.4. Choose a maximal  $K$ -invariant filtration  $\emptyset = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \dots \subset \mathcal{F}_k = \{[F_n]\}$  by free factor systems,

one of which is  $\mathcal{F}_{\text{ng}}$  as defined just before Lemma 5.1.

Assuming  $\mathcal{F}_{k-1} \sqsubset \mathcal{F}_k = \{[F_n]\}$  is a multi-edge extension, since  $K$  is finitely generated we may apply Theorem D of [HM13a] to produce an element  $\theta \in K$  which is fully irreducible relative to the extension  $\mathcal{F}_{k-1} \sqsubset \mathcal{F}_k$ . Choosing a CT representing a rotationless power of  $\theta$  in which  $\mathcal{F}_{\text{ng}}$  and  $\mathcal{F}_{k-1}$  are represented by filtration elements, the highest stratum of that CT must be EG, contradicting Lemma 5.1. It follows that  $\mathcal{F}_{k-1} \sqsubset \mathcal{F}_k = \{[F_n]\}$  is a one-edge extension.

Choose a marked graph  $G$  with a subgraph  $H$  such that  $[H] = \mathcal{F}_{k-1}$  and such that  $G \setminus H$  is a single edge  $E$ . Since  $[H]$  is  $K$ -invariant, each  $\phi \in K$  is represented by a homotopy equivalence  $h : (G, H) \rightarrow (G, H)$  such that  $h(E) = \bar{v}Eu$  for some paths  $u, v \subset H$ . Thus the one-edge free splitting  $\langle G, H \rangle$  is  $K$ -invariant. Since  $K^\phi = K$ , it follows that  $\langle G, H \rangle^{\phi^n}$  is  $K$ -invariant for each  $n \in \mathbb{Z}$ . Applying this to  $\phi^{-n}\psi^n \in K$  we have  $\langle G, H \rangle^{\phi^n} = \langle G, H \rangle^{\phi^n(\phi^{-n}\psi^n)} = \langle G, H \rangle^{\psi^n}$ . Thus the actions of  $\phi$  and  $\psi$  on  $\mathcal{FS}(F_n)$  have a common orbit, and so in the Gromov boundary of  $\mathcal{FS}(F_n)$  the forward limit point of this orbit is  $\partial_+\phi = \partial_+\psi$  and its backward limit point is  $\partial_-\phi = \partial_-\psi$ .

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